

Solving QBF with Counterexample Guided Refinement

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QBF

- an extension of SAT with quantifiers
- PSPACE-complete
- formal verification
- planning

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- we consider **prenex** form with **maximal blocks** of variables

$$QX_1 \bar{Q}Y_1 QX_2 \bar{Q}Y_2 \dots \phi$$

$$\text{where } Q \in \{\exists, \forall\}$$

$$\bar{\exists} = \forall, \bar{\forall} = \exists$$

A QBF as a Game

- it is useful to think about a QBF as a game between the **universal** and **existential player**
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$$\forall y_1 y_2 \exists x_1 x_2. (y_1 \leftrightarrow x_1) \wedge (y_2 \leftrightarrow x_2)$$

- \exists always wins by playing $x_1 = y_1, x_2 = y_2$

Semantics with Winning Move

winning move, **base case** $QX.\phi$, for ϕ propositional

- for $Q = \exists$, an assignment that makes ϕ **true** (model of ϕ)
- for $Q = \forall$, an assignment that makes ϕ **false**

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countermove, for $QX. \Phi$, for Φ QBF

- an assignment μ is a **countermove** to the assignment τ if μ is a winning move for \bar{Q} for $\Phi[\tau]$

Winning Move Semantics

QBF semantics

- $\exists X.\Phi$ is **true** if and only if there is a winning move for \exists
- $\forall X.\Phi$ is **false** if and only if there is a winning move for \forall

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Example

$$\forall y \exists x. x \wedge (y \vee \bar{x})$$

- $\{\bar{y}\}$ is a winning move for \forall , formula is false
- $\{y\}$ is not a winning move and $\{x\}$ is a countermove

Computing a Winning Move—Base Case

Solve ($\exists X. \phi$), where ϕ is a propositional

output : a winning move for \exists if there is one; **NULL**
otherwise

return **SAT**(ϕ)

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Solve ($\forall X. \phi$), where ϕ is a propositional

output : a winning move for \forall if there is one; **NULL**
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return **SAT**($\neg\phi$)

Naive Algorithm for a Winning Move

```
1 Function Solve ( $QX. \Phi$ )
2  $\Lambda \leftarrow \{\text{true}, \text{false}\}^X$            // consider all assignments
3 while true do
4   if  $\Lambda = \emptyset$  then
5     | return NULL           // all assignments used up
6      $\tau \leftarrow \text{pick}(\Lambda)$        // pick a candidate solution
7      $\mu \leftarrow \text{Solve}(\Phi[\tau])$      // find a countermove
8     if  $\mu = \text{NULL}$  then
9       | return  $\tau$            // winning move
10     $\Lambda \leftarrow \Lambda \setminus \{\tau\}$  // remove bad candidate
11 end
```

Removing More Than One Candidate at a Time

Observation

- The naive algorithm does not avail of the countermove

Removing More Than One Candidate at a Time

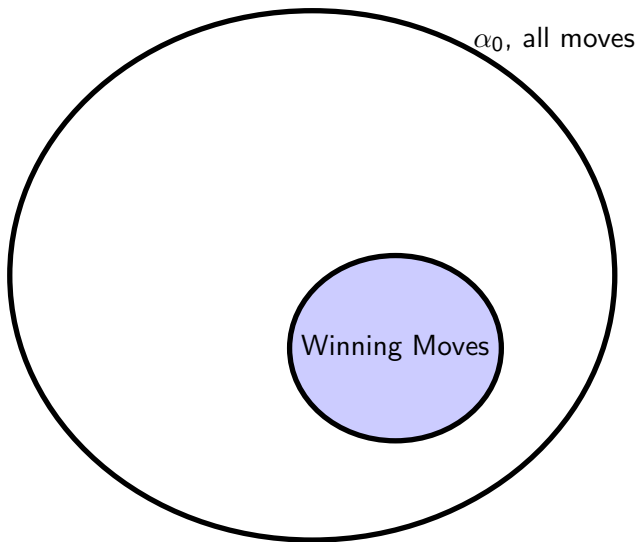
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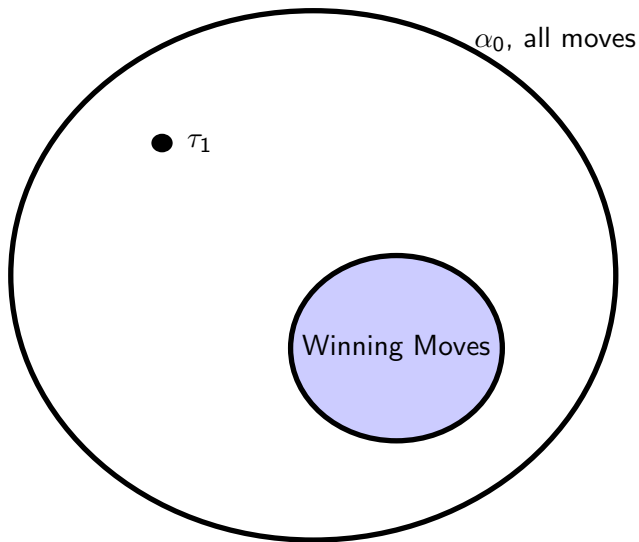
How?

- represent the set of considered candidates as the set of winning moves of a (simpler) QBF (**abstraction**)
- each time a countermove is found, strengthen the abstraction so that the same countermove cannot be used in the future (**refinement**)

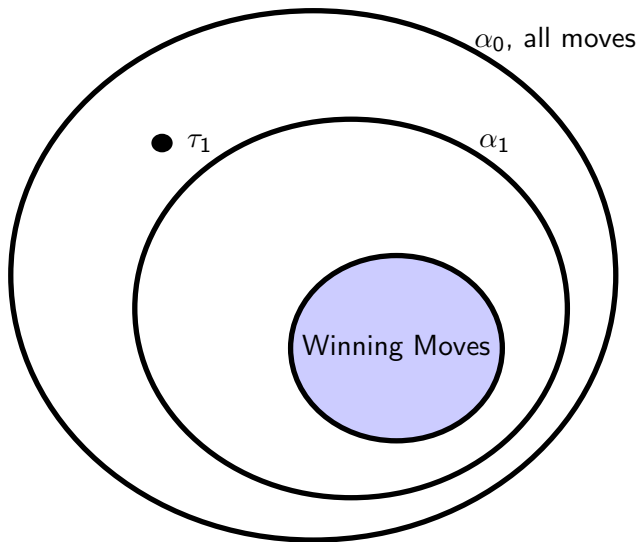
CounterExample Guided Abstraction Refinement (CEGAR)



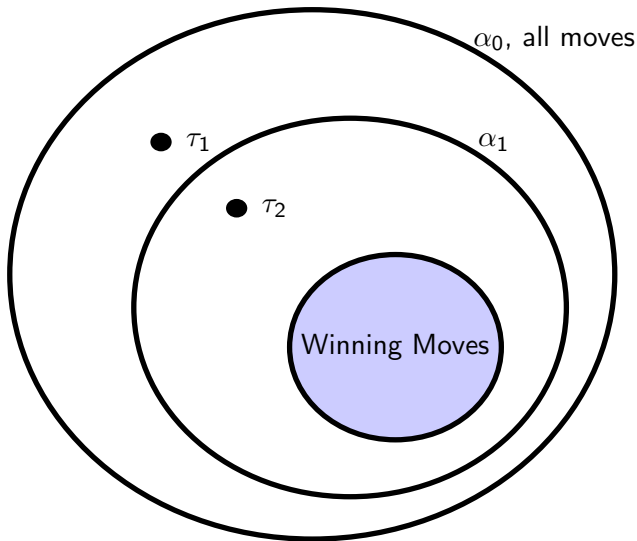
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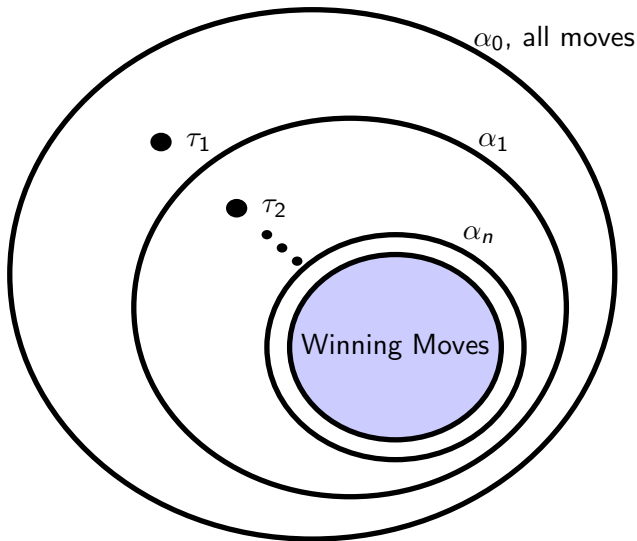
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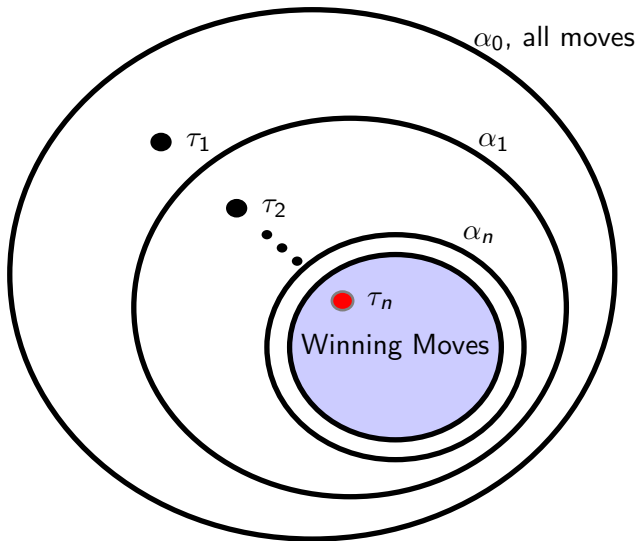
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for a set of countermoves $\omega = \{\mu_1, \dots, \mu_n\}$

- $\bigwedge_{\mu \in \omega} \Phi[\mu], Q = \exists$
- $\bigvee_{\mu \in \omega} \Phi[\mu], Q = \forall$

Refinement—Example

- $\forall y \exists x. x \wedge (y \vee \bar{x})$
- candidate: $\{y\}$, countermove: $\{x\}$
- abstraction: $\forall y. y$
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Abstraction-Based Algorithm for a Winning Move

```
1 Function Solve ( $QX. \Phi$ )
2 begin
3   if  $\Phi$  has no quant then
4     | return ( $Q = \exists$ ) ? SAT( $\phi$ ) : SAT( $\neg\phi$ )
5      $\omega \leftarrow \emptyset$ 
6     while true do
7       |  $\alpha \leftarrow (Q = \exists) ? \bigwedge_{\mu \in \omega} \Phi[\mu] : \bigvee_{\mu \in \omega} \Phi[\mu]$  // abstraction
8       |  $\tau' \leftarrow$  Solve(Prenex( $QX. \alpha$ )) // find a candidate
9       | if  $\tau' = \text{NULL}$  then return NULL // no winning move
10      |  $\tau \leftarrow \{l \mid l \in \tau' \wedge \text{var}(l) \in X\}$  // filter a move for  $X$ 
11      |  $\mu \leftarrow$  Solve( $\Phi[\tau]$ ) // find a countermove
12      | if  $\mu = \text{NULL}$  then return  $\tau$  // winning move
13      |  $\omega \leftarrow \omega \cup \{\mu\}$  // refine
14   end
15 end
```


Results

- **RAReQS** implementation of the above using minisat2.2

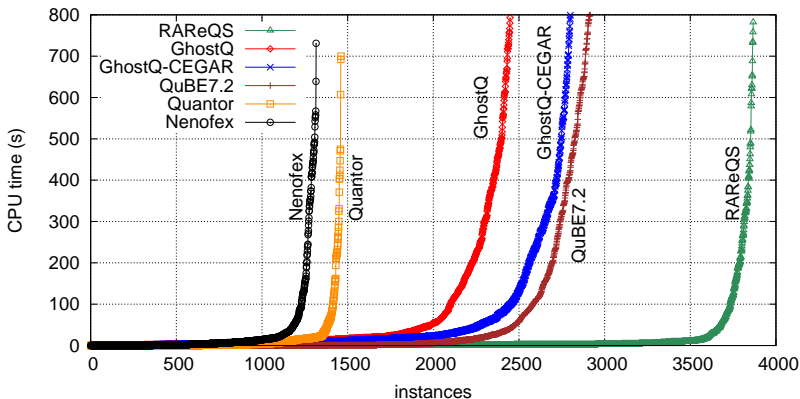
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Results for planning and Formal Verification families



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- enables solving a large number of practical instances not solved by state-of-the-art solvers
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- in the future we plan to further develop the integration between DPLL and CEGAR
- in RAReQS we plan to investigate how to integrate techniques used in other solvers (e.g. dependency detection)

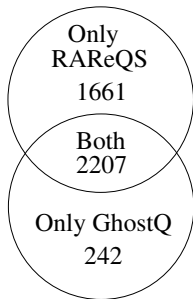
Questions?

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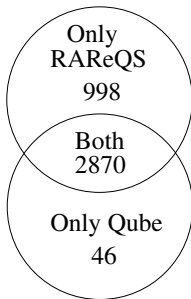
Total instances solved (out of 4669):

RAReQS	GhostQ	GhostQ-C	Qube	Quantor	Nenofex
3868	2449	2801	2916	1462	1317

RAReQS vs GhostQ



RAReQS vs Qube



RAReQS vs Quantor

