Parameterized Complexity of Weighted Satisfiability Problems

Nadia Creignou, Heribert Vollmer

Université d'Aix-Marseille
Leibniz Universität Hannover

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Outline

Weighted Satisfiability

Problem:	p-WSAT/ p-WCIRCUIT-SAT
Input:	a formula/ a circuit φ and $k \in \mathbb{N}$
Parameter:	k
Question:	Does φ have a satisfying assignment of weight exactly k?

$\mathrm{W[SAT]}$-complete for formulas/ $\mathrm{W[P]}$-complete for circuits

Do there exist fragments of lower complexity?

Known results

- The problems remain hard when restricted to the monotone fragment Abrahamson, Downey, Fellows 1995
- A dichotomy in Schaefer's framework for formulas:
  Let S be a finite set of Boolean functions. An S-formula is a conjunction of S-constraints.

Problem:	p-WSAT(S)
Input:	an S-formula φ and $k \in \mathbb{N}$
Parameter:	k
Question:	Does φ have a satisfying assignment of weight exactly k?

Typical problems so captured: 3-SAT, 2-SAT, 1-in-3-SAT, NAE-3-SAT, k-Horn-SAT etc .....

W[1]-complete/FPT Marx 2005

Our framework

n-ary Boolean function $f : \{0,1\}^n \to \{0,1\}$

B a set of Boolean functions = set of allowed connectives

A *B-formula/circuit* is a formula/circuit whose connectives are functions from B

Problem:	p-WSAT(B)/p-WCIRCSAT(B)
Input:	a B-formula/B-circuit φ and $k \in \mathbb{N}$
Parameter:	k
Question:	Does φ have a satisfying assignment of weight exactly k?

Goal: complexity classification of these problems according to B.

fpt-reductions, W[SAT] and W[P]

$R\colon (Q,\kappa) \mapsto (Q',\kappa')$ is an fpt-reduction if for all x:

- $x \in Q$ if and only if $R(x) \in Q'$.
- $R(x)$ is computable by an fpt-algorithm, i.e., in time $f(\kappa(x)) \cdot p(|x|)$.
- $\kappa'(R(x)) \leq g(\kappa(x))$

$$\mathrm{W[SAT]} := [\text{p-WSAT}\{\wedge,\vee,\neg\}]^{\mathrm{fpt}}.$$

$$\mathrm{W[P]} := [\text{p-WCIRCUIT-SAT}\{\wedge,\vee,\neg\}]^{\mathrm{fpt}}.$$

Clones of Boolean functions

Clone: set of Boolean functions closed under superposition,

- contains all projections,
- closed under arbitrary composition

$[B]$ the smallest clone containing B, call B a base for $[B]$

Let $h(x,y) = x \wedge \neg y$.

- $x \wedge y \equiv h(x, h(x,y))$, an $\{h\}$-representation of the function $\wedge$
- $\wedge \in [\{h\}]$

$[B]$ = all Boolean functions that can be computed/represented by B-circuits.

Example

$\mathrm{BF} = [\{\wedge, \vee, \neg\}] = [\{\wedge, \neg\}] = [\{\vee, \neg\}]$

Interest of clones in our study

$x \land y \equiv h(x, h(x, y))$, an $\{h\}$-representation of the function $\land$

$$\land \in [\{h\}]$$

$$\text{p-WCIRCUIT-SAT}(\{\land\}) \leq \text{p-WCIRCUIT-SAT}(\{h\})$$

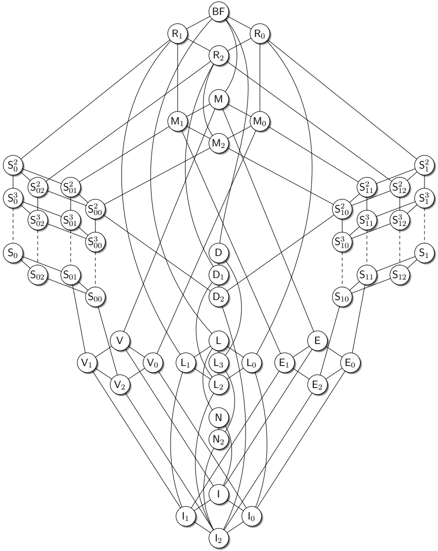
- Given an $\{\land\}$-circuit, replace all gates $\land$ by their $\{h\}$-representation, so you obtain an $\{h\}$-circuit.
- For formulas, this reduction is not necessarily in polynomial time $\longrightarrow$ we need short representations
- Intuition: to get a classification no need to look at all sets B, it is sufficient to consider clones.

$$\text{p-WCIRCUIT-SAT}([B]) \leq \text{p-WCIRCUIT-SAT}(B)$$

Post's lattice

A clone is a set of Boolean functions closed under superposition.
All clones identified by Post in 1941:

- infinitely many, countable
- they form a lattice structure
- every clone has a finite base


BF
R_1
R_0
R_2
M
M_1
M_0
M_2
S_0^2
S_{02}^2
S_{01}^2
S_{00}^2
S_0^3
S_{02}^3
S_{01}^3
S_{00}^3
S_0
S_{02}
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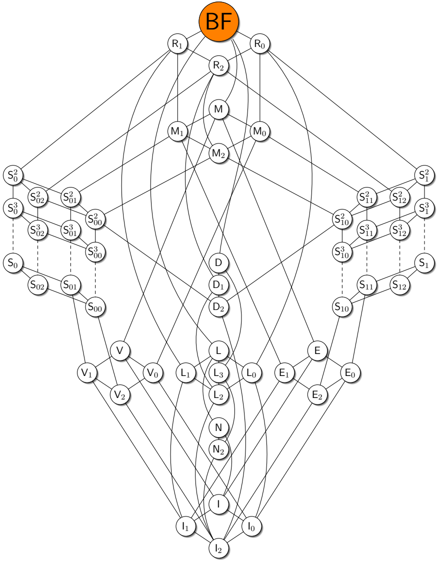
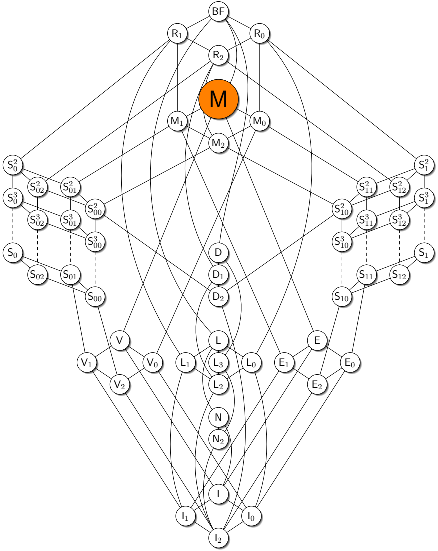
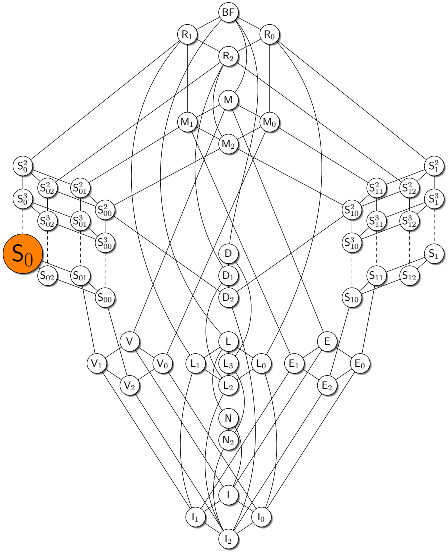


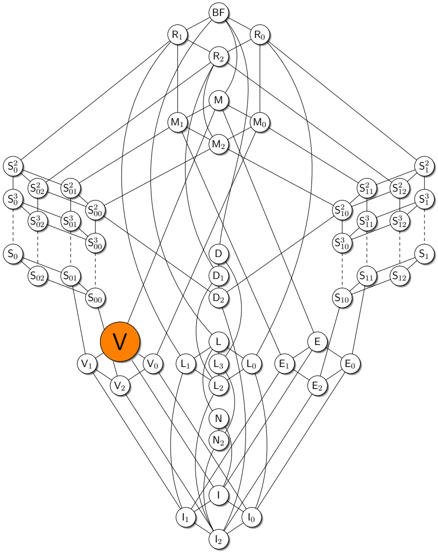
$$BF = [\wedge, \neg] = [\vee, \neg]$$



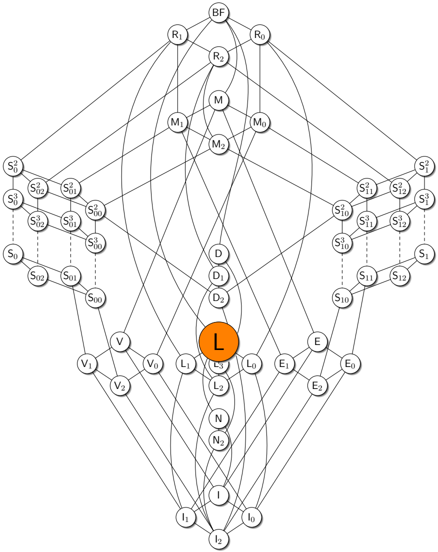
$$\mathrm{BF} = [\wedge, \neg] = [\vee, \neg]$$
$$\mathrm{M} = [\vee, \wedge, 0, 1]$$



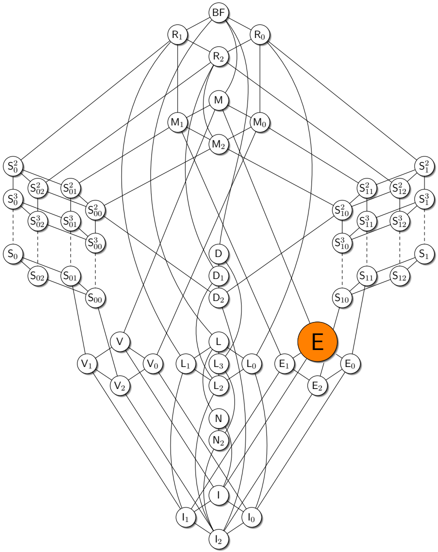
$$\mathrm{BF} = [\wedge, \neg] = [\vee, \neg]$$
$$\mathrm{M} = [\vee, \wedge, 0, 1]$$
$$\mathrm{S}_0 = [\rightarrow]$$



$$\mathsf{BF} = [\wedge, \neg] = [\vee, \neg]$$
$$\mathsf{M} = [\vee, \wedge, 0, 1]$$
$$\mathsf{S}_0 = [\rightarrow]$$
$$\mathsf{V} = [\vee, 0, 1]$$



$\mathrm{BF} = [\wedge, \neg] = [\vee, \neg]$
$\mathrm{M} = [\vee, \wedge, 0, 1]$
$\mathrm{S}_0 = [\rightarrow]$
$\mathrm{V} = [\vee, 0, 1]$
$\mathrm{L} = [\oplus, 0, 1] = [\oplus, 1]$



$$\mathsf{BF} = [\wedge, \neg] = [\vee, \neg]$$
$$\mathsf{M} = [\vee, \wedge, 0, 1]$$
$$\mathsf{S}_0 = [\rightarrow]$$
$$\mathsf{V} = [\vee, 0, 1]$$
$$\mathsf{L} = [\oplus, 0, 1] = [\oplus, 1]$$
$$\mathsf{E} = [\wedge, 0, 1]$$

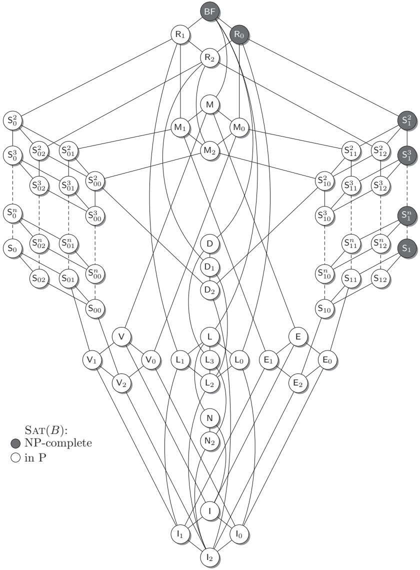
A dichotomy for $\textsc{Sat}(B)$

Problem:	$\textsc{Sat}(B)$
Input:	a B-formula φ
Question:	Is φ satisfiable?

Theorem (Lewis 1979)

$\textsc{Sat}(B)$ is either in P or NP-complete.
It is NP-complete as soon as B can express $\neg(x \rightarrow y)$.

Lewis' result



A dichotomy for p-WSAT(B)

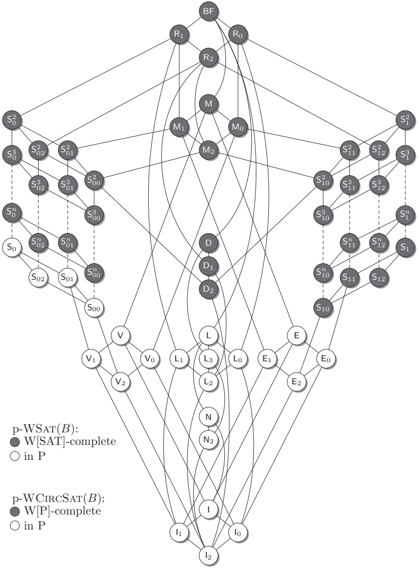
Theorem

p-WSAT(B) *is either in P or* W[SAT]*-complete.*

It is W[SAT]*-complete as soon as B can express either the function $x \wedge (y \vee z)$ or a 2-threshold function.*

A 2-threshold function is a function of the form $f(x_1, \ldots, x_n) = 1$ if and only if $\sum_{i=1}^{n} x_i \geq 2$, for arbitrary n.

A dichotomy for p-WSAT(B)



Sketch of proof (1)

- Membership follows (almost) by definition
- p-WSAT(M) is known to be W[SAT]-hard
- If $D_2 \subseteq [B]$ or $S_{10} \subseteq [B]$ or $S_{00}^n \subseteq [B]$ for some $n \geq 2$, then

$$\text{p-WSAT}(M) \leq^{\text{fpt}} \text{p-WSAT}(B \cup \{0,1\}) \leq^{\text{fpt}} \text{p-WSAT}(B).$$

Sketch of proof (2): why threshold functions?

$$\text{p-WS}{\small\text{AT}}(\mathrm{M}) \leq^{\text{fpt}} \text{p-WS}{\small\text{AT}}(B \cup \{0,1\}) \leq^{\text{fpt}} \text{p-WS}{\small\text{AT}}(B).$$

the difficulty is to mimic the rôle of the constant 0

Definition

A formula is l-costly if all its satisfying assignment have weight at least l

q-threshold functions ($q \geq 2$) allow to represent $(k+1)$- costly functions

Sketch of proof (3)

- If $[B] \subseteq \mathrm{V}, \mathrm{E}$, or L, then every function computed by a B-circuit has a normal form, which is easy to compute and from which k-satisfiability is easy to decide.
- If $[B] \subseteq \mathrm{S}_{00}$ then every B-circuit can be tranformed into an $\{\rightarrow\}$-circuit, from which k-satisfiability is easy to decide.

$\mathrm{p}\text{-}\#\text{WSAT}\ (B)$

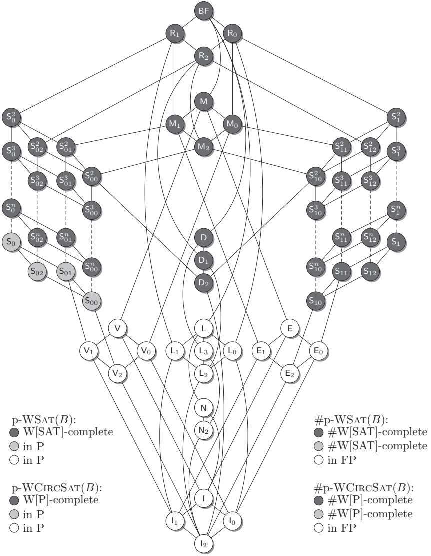
Problem:	$\mathrm{p}\text{-}\#\text{WSAT}\ (B)$
Input:	a B-formula φ and $k \in \mathbb{N}$
Parameter:	k
Output:	Number of satisfying assignments for φ of weight exactly k

Definition

The class $\#\mathrm{W[SAT]}$ is the closure of $\mathrm{p}\text{-}\#\text{WSAT}\ (\{\wedge, \vee, \neg\})$ under fpt-parsimonious reductions, that is

$$\#\mathrm{W[SAT]} := [\mathrm{p}\text{-}\#\text{WSAT}\ (\{\wedge, \vee, \neg\})]^{\mathrm{fpt}}.$$

Complexity classifications at a glance



Counting versus decision

1. All hard decision problems become hard counting problems
2. p-WSAT$(\{\rightarrow\})$ is in P, whereas p-$\#$WSAT $(\rightarrow)$ is $\#\mathrm{W}[\mathrm{SAT}]$-complete.
3. Hardness results are obtained under Turing fpt-reductions.

Sketch of proof

- p-#WSAT (M_2) is known to be $\#\mathrm{W[SAT]}$-hard
- If $\mathsf{D}_2, \mathsf{S}_{10}$ or $\mathsf{S}_{00} \subseteq [B]$, then either

  $$\text{p-\#WSAT}\ (\mathsf{M}_2) \leq^{\text{fpt-T}} \text{p-\#WSAT}\ (B \cup \{0\}) \leq^{\text{fpt-T}} \text{p-\#WSAT}\ (B),$$

  or

  $$\text{p-\#WSAT}\ (\mathsf{M}_2) \leq^{\text{fpt-T}} \text{p-\#WSAT}\ (B \cup \{1\}) \leq^{\text{fpt-T}} \text{p-\#WSAT}\ (B),$$

  Hardness results are obtained through Turing fpt-reductions
- Easy cases follow from an appropriate normal form.

Conclusion

- Complete classification of the parameterized complexity of p-WSAT(B) and p-WCIRCUIT-SAT(B) (and of their counting counterparts)
- pinpoint the reason for intractability of weighted satisfiability by exhibiting which Boolean functions make the problem hard.
- a little disappointing not to see any involved FPT algorithm
- While the class $\#\mathrm{W[P]}$ was introduced in Flum, Grohe 2004 we here introduced $\#\mathrm{W[SAT]}$, and we present natural complete satisfiability problems for both classes.
- future work: satisfying assignments of weight $\geq k$ or $\leq k$.