

Parameterized Complexity of Weighted Satisfiability Problems

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Outline

- 1 Introduction and Goal
- 2 Preliminaries
 - Parameterized complexity
 - Clones and Post's lattice
 - Lewis' result
- 3 Results
 - Parameterized decision problems
 - Parameterized counting problems

Weighted Satisfiability

Problem: p-WSAT/ p-WCIRCUIT-SAT

Input: a formula/ a circuit φ and $k \in \mathbb{N}$

Parameter: k

Question: Does φ have a satisfying assignment of weight exactly k ?

W[SAT]-complete for formulas/ W[P]-complete for circuits

Do there exist fragments of lower complexity?

Known results

- The problems remain hard when restricted to the monotone fragment **Abrahamson, Downey, Fellows 1995**
- A dichotomy in Schaefer's framework for formulas:
Let S be a finite set of Boolean functions. An S -formula is a conjunction of S -constraints.

Problem: p-WSAT(S)

Input: an S -formula φ and $k \in \mathbb{N}$

Parameter: k

Question: Does φ have a satisfying assignment of weight exactly k ?

Typical problems so captured: 3-SAT, 2-SAT, 1-in-3-SAT, NAE-3-SAT, k -Horn-SAT etc

W[1]-complete/FPT Marx 2005

Our framework

n -ary Boolean function $f : \{0, 1\}^n \rightarrow \{0, 1\}$

B a set of Boolean functions = set of allowed connectives

A B -formula/circuit is a formula/circuit whose connectives are functions from B

Problem: p -WSAT(B)/ p -WCIRCSAT(B)

Input: a B -formula/ B -circuit φ and $k \in \mathbb{N}$

Parameter: k

Question: Does φ have a satisfying assignment of weight exactly k ?

Goal: complexity classification of these problems according to B .

fpt-reductions, W[SAT] and W[P]

$R: (Q, \kappa) \mapsto (Q', \kappa')$ is an **fpt-reduction** if for all x :

- $x \in Q$ if and only if $R(x) \in Q'$.
- $R(x)$ is computable by an fpt-algorithm, i.e., in time $f(\kappa(x)) \cdot p(|x|)$.
- $\kappa'(R(x)) \leq g(\kappa(x))$

$$W[\text{SAT}] := [\text{p-WSAT}\{\wedge, \vee, \neg\}]^{\text{fpt}}.$$

$$W[\text{P}] := [\text{p-WCIRCUIT-SAT}\{\wedge, \vee, \neg\}]^{\text{fpt}}.$$

Clones of Boolean functions

Clone: set of Boolean functions closed under superposition,

- contains all projections,
- closed under arbitrary composition

$[B]$ the smallest clone containing B , call B a **base** for $[B]$

Let $h(x, y) = x \wedge \neg y$.

- $x \wedge y \equiv h(x, h(x, y))$, an $\{h\}$ -representation of the function \wedge
- $\wedge \in [\{h\}]$

$[B]$ = all Boolean functions that can be computed/represented by B -circuits.

Example

$$\text{BF} = [\{\wedge, \vee, \neg\}] = [\{\wedge, \neg\}] = [\{\vee, \neg\}]$$

Interest of clones in our study

$x \wedge y \equiv h(x, h(x, y))$, an $\{h\}$ -representation of the function \wedge

$$\wedge \in [\{h\}]$$

$$\text{p-WCIRCUIT-SAT}(\{\wedge\}) \leq \text{p-WCIRCUIT-SAT}(\{h\})$$

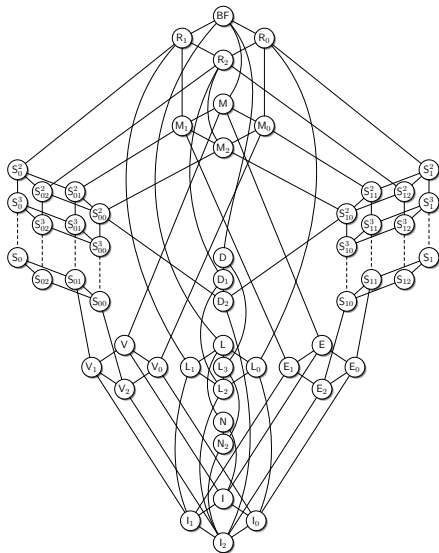
- Given an $\{\wedge\}$ -circuit, replace all gates \wedge by their $\{h\}$ -representation, so you obtain an $\{h\}$ -circuit.
- For formulas, this reduction is not necessarily in polynomial time \rightarrow we need **short representations**
- **Intuition**: to get a classification no need to look at all sets B , it is sufficient to consider clones.

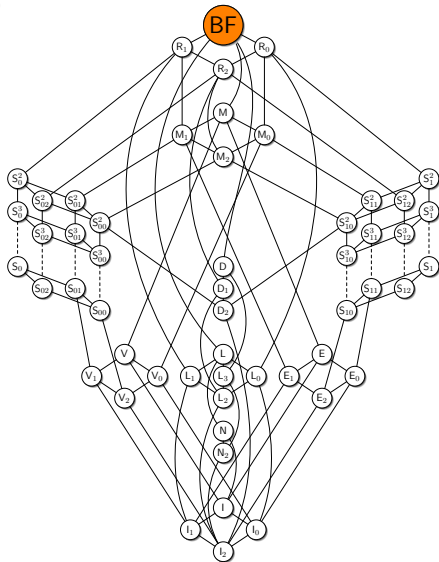
$$\text{p-WCIRCUIT-SAT}([B]) \leq \text{p-WCIRCUIT-SAT}(B)$$

Post's lattice

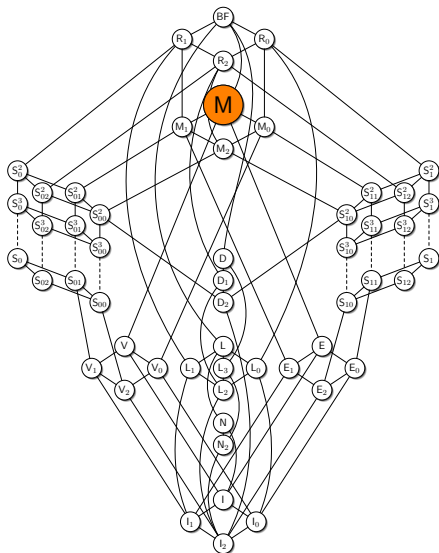
A clone is a set of Boolean functions closed under superposition.
All clones identified by Post in 1941:

- infinitely many, countable
- they form a lattice structure
- every clone has a finite base



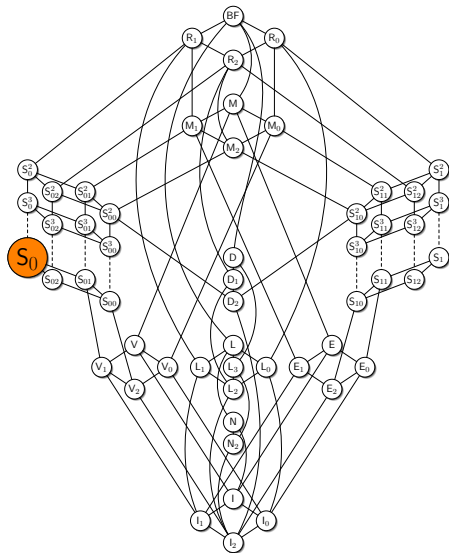


$$\mathbf{BF} = [\wedge, \neg] = [\vee, \neg]$$



$$\text{BF} = [\wedge, \neg] = [\vee, \neg]$$

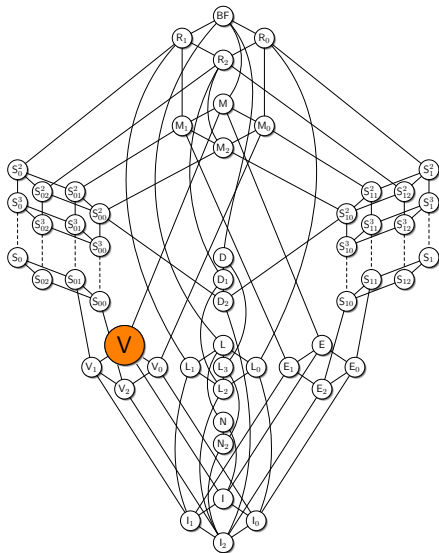
$$\text{M} = [\vee, \wedge, 0, 1]$$



$$\text{BF} = [\wedge, \neg] = [\vee, \neg]$$

$$\text{M} = [\vee, \wedge, \mathbf{0}, \mathbf{1}]$$

$$\text{S}_0 = [\rightarrow]$$

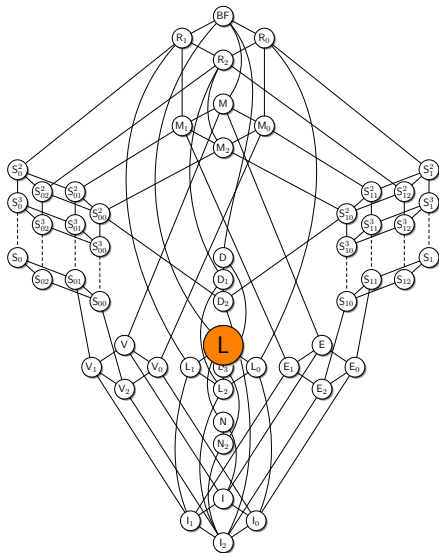


$$BF = [\wedge, \neg] = [V, \neg]$$

$$M = [V, \wedge, 0, 1]$$

$$S_0 = [\rightarrow]$$

$$V = [V, 0, 1]$$



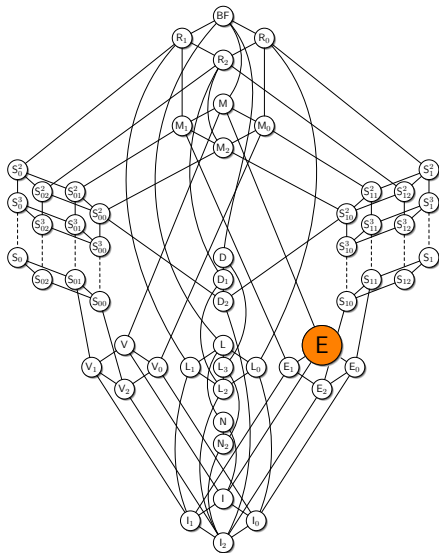
$$\mathbf{BF} = [\wedge, \neg] = [\vee, \neg]$$

$$\mathbf{M} = [\vee, \wedge, \mathbf{0}, \mathbf{1}]$$

$$\mathbf{S}_0 = [\rightarrow]$$

$$\mathbf{V} = [\vee, \mathbf{0}, \mathbf{1}]$$

$$\mathbf{L} = [\oplus, \mathbf{0}, \mathbf{1}] = [\oplus, \mathbf{1}]$$



$$BF = [\wedge, \neg] = [\vee, \neg]$$

$$M = [\vee, \wedge, 0, 1]$$

$$S_0 = [\rightarrow]$$

$$V = [\vee, 0, 1]$$

$$L = [\oplus, 0, 1] = [\oplus, 1]$$

$$E = [\wedge, 0, 1]$$

A dichotomy for $\text{SAT}(B)$

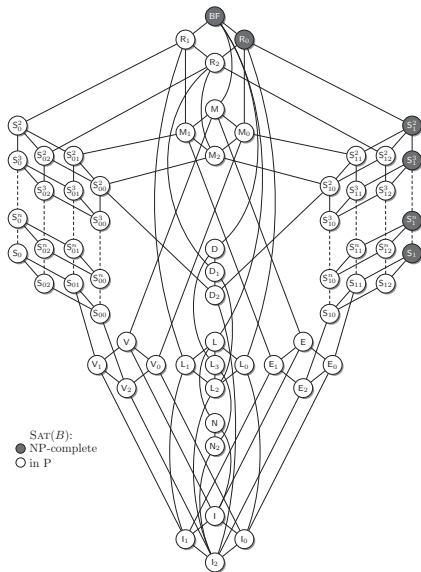
Problem: $\text{SAT}(B)$
Input: a B -formula φ
Question: Is φ satisfiable?

Theorem (Lewis 1979)

$\text{SAT}(B)$ is either in P or NP -complete.

It is NP -complete as soon as B can express $\neg(x \rightarrow y)$.

Lewis' result



A dichotomy for p -WSAT(B)

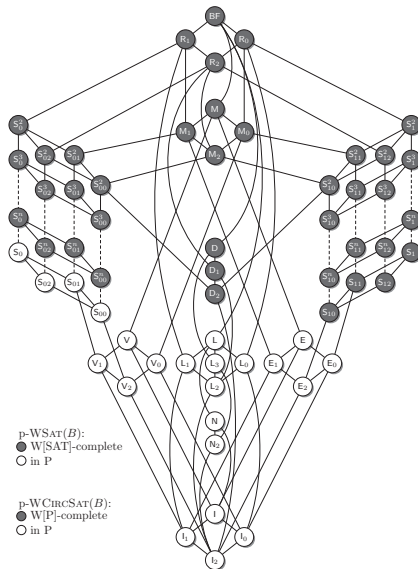
Theorem

p -WSAT(B) is either in P or W[SAT]-complete.

It is W[SAT]-complete as soon as B can express either the function $x \wedge (y \vee z)$ or a 2-threshold function.

A **2-threshold function** is a function of the form $f(x_1, \dots, x_n) = 1$ if and only if $\sum_{i=1}^n x_i \geq 2$, for arbitrary n .

A dichotomy for $p\text{-WSAT}(B)$



Sketch of proof (1)

- Membership follows (almost) by definition
- $p\text{-WSAT}(M)$ is known to be $W[\text{SAT}]$ -hard
- If $D_2 \subseteq [B]$ or $S_{10} \subseteq [B]$ or $S_{00}^n \subseteq [B]$ for some $n \geq 2$, then

$$p\text{-WSAT}(M) \leq^{\text{fpt}} p\text{-WSAT}(B \cup \{0, 1\}) \leq^{\text{fpt}} p\text{-WSAT}(B).$$

Sketch of proof (2): why threshold functions?

$$\text{p-WSAT}(M) \leq^{\text{fpt}} \text{p-WSAT}(B \cup \{0, 1\}) \leq^{\text{fpt}} \text{p-WSAT}(B).$$

the difficulty is to mimic the rôle of the constant 0

Definition

A formula is *l-costly* if all its satisfying assignment have weight at least l

q-threshold functions ($q \geq 2$) allow to represent $(k + 1)$ - costly functions

Sketch of proof (3)

- If $[B] \subseteq V, E, \text{ or } L$, then every function computed by a B -circuit has a normal form, which is easy to compute and from which k -satisfiability is easy to decide.
- If $[B] \subseteq S_{00}$ then every B -circuit can be transformed into an $\{\rightarrow\}$ -circuit, from which k -satisfiability is easy to decide.

$p\text{-}\#\text{WSAT}(B)$

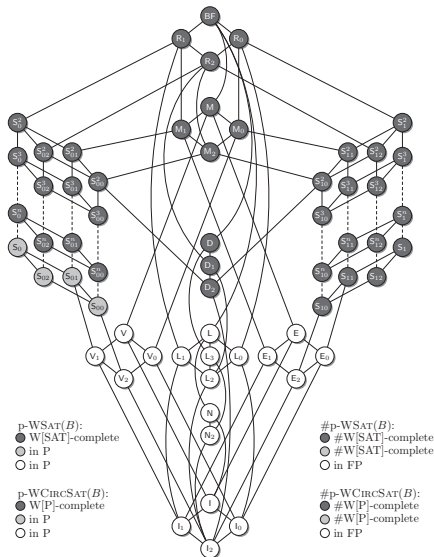
- Problem:* $p\text{-}\#\text{WSAT}(B)$
Input: a B -formula φ and $k \in \mathbb{N}$
Parameter: k
Output: Number of satisfying assignments for φ of weight exactly k

Definition

The class $\#\text{W}[\text{SAT}]$ is the closure of $p\text{-}\#\text{WSAT}(\{\wedge, \vee, \neg\})$ under fpt -parsimonious reductions, that is

$$\#\text{W}[\text{SAT}] := [p\text{-}\#\text{WSAT}(\{\wedge, \vee, \neg\})]^{\text{fpt}}.$$

Complexity classifications at a glance



Counting versus decision

- 1 All hard decision problems become hard counting problems
- 2 $p\text{-WSAT}(\{\rightarrow\})$ is in P, whereas $p\text{-}\#\text{WSAT}(\rightarrow)$ is $\#\text{W}[\text{SAT}]$ -complete.
- 3 Hardness results are obtained under Turing f_{pt} -reductions.

Sketch of proof

- $p\text{-}\#\text{WSAT}(M_2)$ is known to be $\#\text{W}[\text{SAT}]$ -hard
- If D_2, S_{10} or $S_{00} \subseteq [B]$, then either

$$p\text{-}\#\text{WSAT}(M_2) \leq^{\text{fpt-T}} p\text{-}\#\text{WSAT}(B \cup \{0\}) \leq^{\text{fpt-T}} p\text{-}\#\text{WSAT}(B),$$

or

$$p\text{-}\#\text{WSAT}(M_2) \leq^{\text{fpt-T}} p\text{-}\#\text{WSAT}(B \cup \{1\}) \leq^{\text{fpt-T}} p\text{-}\#\text{WSAT}(B),$$

Hardness results are obtained through Turing fpt -reductions

- Easy cases follow from an appropriate normal form.

Conclusion

- Complete classification of the parameterized complexity of $p\text{-WSAT}(B)$ and $p\text{-WCIRCUIT-SAT}(B)$ (and of their counting counterparts)
- pinpoint the reason for intractability of weighted satisfiability by exhibiting which Boolean functions make the problem hard.
- a little disappointing not to see any involved FPT algorithm
- While the class $\#W[P]$ was introduced in [Flum, Grohe 2004](#) we here introduced $\#W[\text{SAT}]$, and we present natural complete satisfiability problems for both classes.
- future work: satisfying assignments of weight $\geq k$ or $\leq k$.