Fixed-parameter tractability of satisfying beyond the number of variables

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Outline of the Talk

- Introduction
  - Fixed-parameter tractability
  - Some parameterized questions, results related to SAT
  - Above guarantee parameterizations
- The main question and the result in the paper
  - Consequences (corollaries)
  - The Algorithm outline
- Conclusions.

Fixed Parameter Tractable (FPT) Algorithms

- Input x comes with a parameter k (e.g. solution size, backdoor size to Horn)
- Solve (x, k) in $f(k) + |x|^{O(1)}$ time where f is a function of k alone.
- Parameters can be anything that makes sense in theory and practice (We will see examples shortly).
- Problems that have such an algorithm are said to be fixed parameter tractable (FPT).
- The problem is said to be in XP if it has an $|x|^{f(k)}$ algorithm.
- There is a W-hardness theory to identify problems that are unlikely to have such algorithms

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Kernelization

- *Kernelization*: Reduce (x, k) in polynomial time to an equivalent instance (x', k') such that $|x'|$ and k' are some functions of k.
- *Folklore*: Fixed-parameter tractable if and only if kernelizable.
- *Interesting question*: When is $|x'|$ a polynomial function of k?
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Parameterized questions/answers related to SAT

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- Is there an assignment satisfying at least k clauses (of the formula wth m clauses)?

  Trivially FPT, as if $k \leq m/2$, the answer is YES, and otherwise, $m \leq 2k$, and we have a kernel for the number of clauses (and one can obtain a kernel for the number of variables with some effort).

- Hence trivial if k is small, and so FPT algorithm applies only when k is large, and the c^k algorithm may not be practical.

- *More appropriate question/parameterization*: Is there an assignment satisfying at least $m/2 + k$ clauses? FPT (Mahajan and Raman, 1999).

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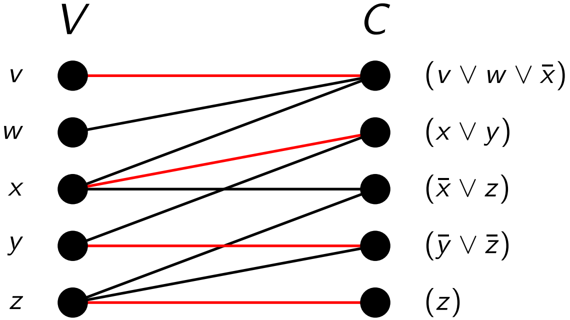

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- Clearly at least μ clauses can be satisfied.
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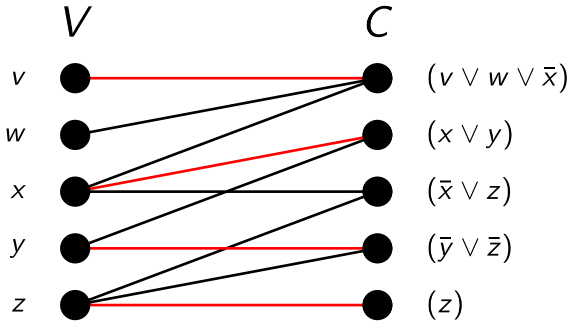

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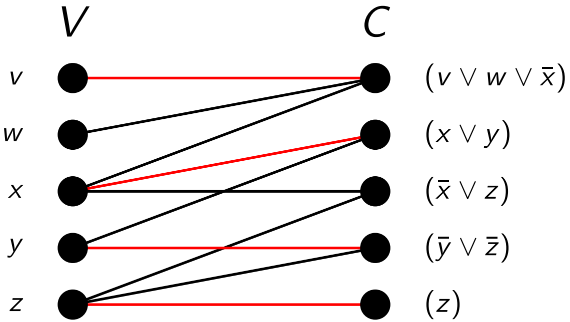

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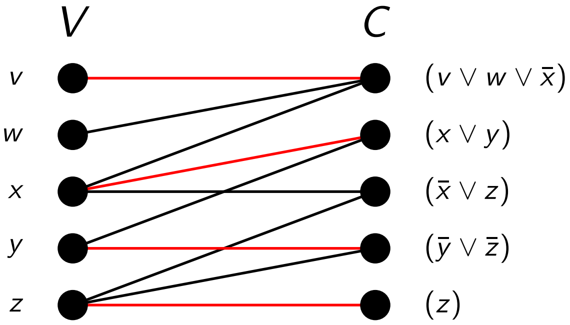

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The consequences (Related work)

- For a *variable-matched* formula (autarky/Crown reduced $\mu = n$), it is FPT to decide whether $n + k$ clauses are satisfiable.

- *Minimal unsatisfiable formulae*
  - are in general hard to recognize (D^p-complete – Papadimitriou and Wolfe);
  - have no crowns, and hence are variable matched (and so one can satisfy at least n clauses)
  - can be recognized in polynomial time if they have $n + k$ clauses (asked by Kleine Buning, shown XP by Kullman, Fleischner and Szeider, FPT by Szeider)
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The Main steps of our algorithm

- Some preprocessing rules
- Branching rules
- Reduce to the $m-k$ hitting set problem (Given a family of m sets, hit all of them with a subset of at most $m-k$ elements; k is the parameter)

The Challenges/Main ideas/Contributions

- Carefully analyzing the drop in (*solnsize* − *matching size*);
- argue that they don't increase in preprocessing steps, and decrease in each branching step.
- Hence need stronger preprocessing rules.
- New (deterministic and improved randomized) algorithms for $(m-k)$ hitting set problem

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- P1. If a variable appears only in pure or only in negated form, set it appropriately.
- P2. (Resolution of variables with exactly two occurrences). If a variable x appears once positively and once negatively, then replace the two clauses with their union after deleting x (and $\bar{x}$).

$$(\mathbf{x} \vee y) \wedge (\bar{\mathbf{x}} \vee z)$$
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At this point, every variable appears at least three times.

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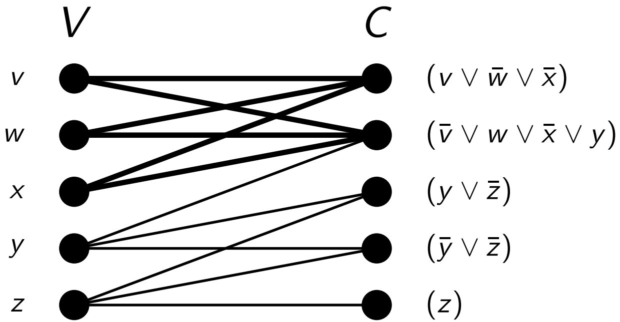

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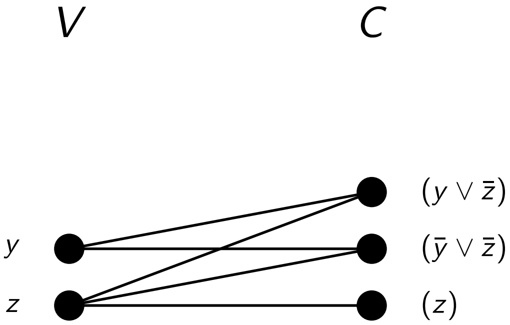
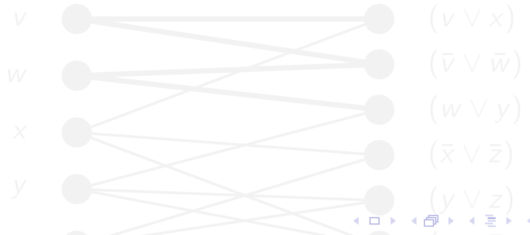


Main preprocessing rule

P4. Let there be a subset S of variables such that $|N(S)| = |S| + 1$. Test whether the clauses of $N(S)$ can be satisfied using variables in S alone.

(Such a subset can be found in polynomial time using matching theory, and sat can be checked in polynomial time using the algorithm for $n+1$ clauses satisfiability.)

- If the clauses in $N(S)$ are satisfiable (with variables in S alone), then remove S and its neighbors.
- Else remove variables in S and clauses in $N(S)$ and add a clause that contains all variables in the clauses in $N(S)$ which are not in S.



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P4. Let there be a subset S of variables such that $|N(S)| = |S| + 1$. Test whether the clauses of $N(S)$ can be satisfied using variables in S alone. (Such a subset can be found in polynomial time using matching theory, and sat can be checked in polynomial time using the algorithm for $n + 1$ clauses satisfiability.)

- If the clauses in $N(S)$ are satisfiable (with variables in S alone), then remove S and its neighbors.
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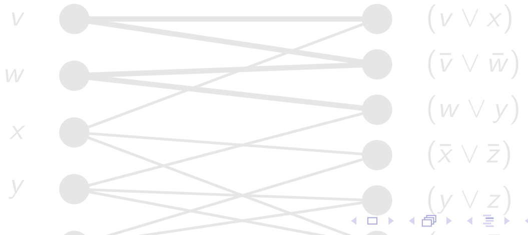

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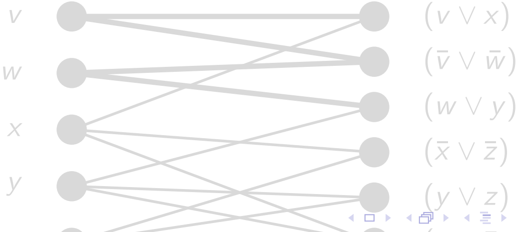

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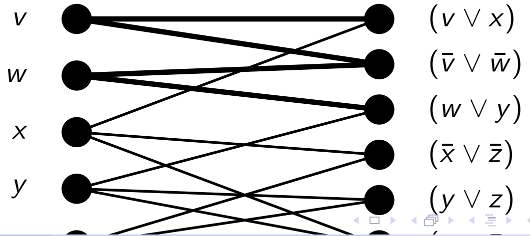

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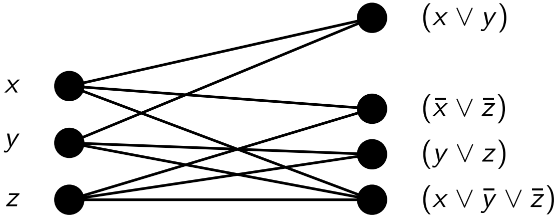

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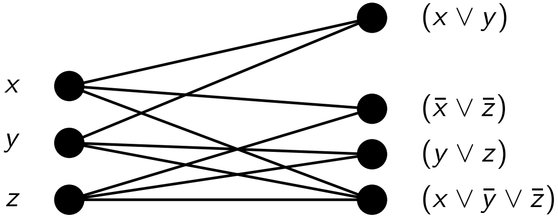

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Branching Rules

- B1. If x and $\bar{x}$ appear at least twice in F, branch by setting $x = 1$ or $x = 0$.

  Key argument: While k, the number to be satisfied, drops by at least $n(x)$, the matching size drops by at most $n(x) - 1$ as the bipartite graph is 2-expanding (i.e. every subset has a buffer of size at least 2).

  Without loss of generality, let $n(x) = 1, n(\bar{x}) > 1$ for every variable x (Otherwise, rename by flipping).

- B2. If a clause contains two positive literals, then branch by setting each of them to 0

  As $n(x) = 1$, this clause is the only clause with positive occurrence for both the variables, so one of them can be false.

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Where are we?

Every clause has at most one positive literal, and every variable has only one positive occurrence.

- If the clause containing x is (x, C), then replace it by (x) and by adding C to all clauses containing $\bar{x}$.

$$(x \vee C) \wedge (\bar{x} \vee \bar{y}) \wedge (\bar{x} \vee \bar{z}) \wedge (\bar{x} \vee \bar{w})$$
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- Resulting formula has a unit clause for each variable, followed by all negated clauses.

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Summary of the Algorithm

- P1. Set variables appearing in one form (pure or negated)
- P2. Resolve variables appearing once positively and once negatively.
- P3. Reduce autarkies
- P4. Reduce subsets S of variables, where $|N(S)| = |S| + 1$.
- B1. If a variable appears at least twice positively and negatively, branch.
- B2. If a clause has two positive literals, branch by setting each of them to false.
- Reduce to $m - k$ hitting set problem after some transformation.

FPT algorithm for $(m - k)$ hitting set

- (Gutin et al): The family F has a $m - k$ hitting set if and only if there is some subset S of the universe such that $|S| \leq k$ and S 'hits' at least $|S| + k$ sets.
- Find such a subset (for each size p up to k), if exists, by color coding:
  - Color the sets randomly using $p + k$ colors
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Conclusions

- Showed FPT algorithm for deciding whether a SAT formula has an assignment satisfying at least $\mu + k$ clauses.
- We also show that the problem has no polynomial in k kernel (under complexity theoretic conditions). By a parameter preserving reduction from $m - k$-hitting set (which has been already shown to have no polynomial kernel under the same conditions)
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Thank You