On Davis-Putnam reductions for minimally unsatisfiable clause-sets

Oliver Kullmann[1] Xishun Zhao[2]

[1]Computer Science, Swansea University, UK

[2]Institute of Logic and Cognition, Sun Yat-sen University, China

SAT 2012, June 20, 2012

Two sources: DP and MU

- Under what circumstances is DP-reduction (single steps of the DP-reduction from Davis and Putnam [2]; also called "variable elimination") confluent or confluent modulo isomorphism?
- Further steps towards the classification of all minimally unsatisfiable clause-sets.

Bibliographical remarks

- Underlying paper is Kullmann and Zhao [9].
- The (extended) underlying technical report is Kullmann and Zhao [10] (containing also some technical corrections).

Outline

Clause-sets

- Literals are variables v and their complements $\overline{v}$.
- A clause C is a finite and clash-free set of literals, i.e., $C \cap \overline{C} = \emptyset$.
- A **clause-set** is a finite set of clauses.
- The set of all clause-sets is $\mathcal{CLS}$.

For example

$$\mathcal{F}_2 = \{ \{v_1, v_2\}, \{\overline{v_1}, \overline{v_2}\}, \{\overline{v_1}, v_2\}, \{\overline{v_2}, v_1\} \}$$

is a clause-set.
Remark: Clause-sets are considered as (precise) combinatorial objects, as generalised hypergraphs.

Applying partial assignments

The application of a partial assignment $\varphi \in \mathcal{PASS}$ to a clause-set $F \in \mathcal{CLS}$ is denoted by

$$\boldsymbol{\varphi * F} \in \mathcal{CLS}.$$

Satisfied clauses are removed, then falsified literals.
A special clause-set is $\top := \emptyset$, a special clause is $\bot := \emptyset$.

- F is **satisfiable** iff there is $\varphi \in \mathcal{PASS}$ with $\varphi * F = \top$.
- $\top$ is satisfiable.
- $\{\bot\}$ is unsatisfiable.
- More generally, every F with $\bot \in F$ is unsatisfiable.

$$\mathcal{CLS} = \mathcal{SAT} \,\dot\cup\, \mathcal{USAT}.$$

Resolution and DP-reduction

Clauses C, D are **resolvable** if $C \cap \overline{D} = \{x\}$:

$$\mathbf{C \diamond D} := (C \setminus \{x\}) \cup (D \setminus \{\overline{x}\}).$$

DP-reduction (or "variable elimination"):

$$\mathbf{DP_v(F)} := \{C \in F : v \notin \mathrm{var}(C)\} \cup \\ \{\, C \diamond D : C, D \in F \wedge C \cap \overline{D} = \{v\}\,\}.$$

$\mathrm{DP}_v(F)$ is semantically the existential quantification of F by v:

$$\mathrm{DP}_v(F) \longleftrightarrow \exists\, v : F.$$

Singular variables

A variable v is **singular** for F if

- in one sign it occurs only once,
- while in the other sign it occurs at least once.

Singular DP-reduction ("sDP-reduction") is $F \rightsquigarrow \mathrm{DP}_v(F)$ for a singular variable.

sDP-reduction decreases the number of clauses at least by one.

TOC: MU and its structures

- MU, SMU
- deficiency
- MU(1)
- non-singularity
- MU(2)

See Kleine Büning and Kullmann [5].

MU and SMU

Minimal unsatisfiability: no clause can be removed.

$$\boldsymbol{\mathcal{MU}} := \{F \in \mathcal{USAT} \mid \forall\, C \in F : F \setminus \{C\} \in \mathcal{SAT}\}.$$

Saturated minimal unsatisfiability: no literal can be added.

$$\boldsymbol{\mathcal{SMU}} := \{F \in \mathcal{MU} \mid \forall\, C \in F\ \forall\, C' \supset C : (F \setminus \{C\}) \cup \{C'\} \in \mathcal{SAT}\}.$$

Lemma

$F \in \mathcal{SMU}$ iff for all $v \in \mathrm{var}(F)$ and $\varepsilon \in \{0,1\}$ holds $\langle v \to \varepsilon \rangle * F \in \mathcal{MU}$.

Deficiency

The basic **complexity parameter** of $F \in \mathcal{MU}$ is the **deficiency**:

$$\begin{aligned} \boldsymbol{\delta(\mathbf{F})} &:= c(F) - n(F) \\ c(F) &:= |F| \\ n(F) &:= |\mathrm{var}(F)|. \end{aligned}$$

The basic fact:

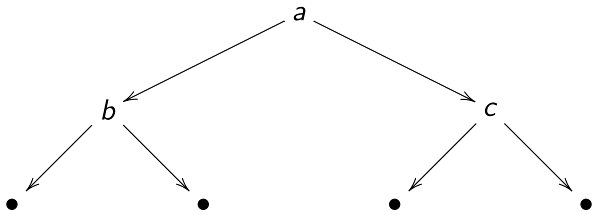
$$F \in \mathcal{MU} \Longrightarrow \delta(F) \geq 1.$$

We use here $\boldsymbol{\mathcal{MU}(\mathbf{k})} := \{F \in \mathcal{MU} : \delta(F) = k\}$.

MU(1)

Timeline:

- Aharoni and Linial [1] (1986) $\mathcal{SMU}(1)$
- Davydov, Davydova, and Kleine Büning [3] (1998) $\mathcal{MU}(1)$
- Kullmann [6] (2000) tree model



1. $\{\{a,b\},\{a,\overline{b}\},\{\overline{a},c\},\{\overline{a},\overline{c}\}\} \in \mathcal{SMU}(1)$. Singular: b, c.
2. $\{\{a,b\},\{\overline{b}\},\{\overline{a},c\},\{\overline{a},\overline{c}\}\} \in \mathcal{MU}(1)$. Singular: all.
3. $\{\{a,b\},\{\overline{b}\},\{\overline{a},c\},\{\overline{c}\}\} \in \mathcal{MMU}(1)$.

Eliminating trivialities

We consider sDP-reduction as eliminating trivialities.

- So all of $\mathcal{MU}(1)$ boils down to $\{\bot\}$.
- Intuitively one can understand applying sDP as removing some "MU(1)-hunch".

A clause-set F is called **non-singular** if it does not contain singular variables.

- The set of singular variables of F is $\mathbf{var_s(F)} \subseteq \mathrm{var}(F)$.
- So F is nonsingular iff $\mathrm{var_s}(F) = \emptyset$.

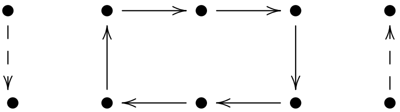
$$\boldsymbol{\mathcal{MU}'} := \{F \in \mathcal{MU} : \mathrm{var_s}(F) = \emptyset\}.$$

MU(2)

A breakthrough was achieved by Kleine Büning [4] (2000).
The elements of $\mathcal{MU}'(2) = \mathcal{SMU}(2)$ are **precisely** the following clause-sets for $n \geq 2$:

$$x_1 \to x_2,\ x_2 \to x_3, \dots, x_{n-1} \to x_n,\ x_n \to x_1$$
$$\{x_1, \dots, x_n\},$$
$$\{\overline{x_1}, \dots, \overline{x_n}\}.$$

That is, one cycle, with opposed forced "directions" $(n = 6)$:



Understanding UNSAT — understanding MU

For unsatisfiable $F \in \mathcal{USAT}$

we want to **"understand"** its unsatisfiability.

We want to SEE it.

Each $F' \subseteq F$ with $F' \in \mathcal{MU}$ contains one explanation:

- The additional clauses in $F \setminus F'$ can make the contradiction of F' more easily accessible, but do not contribute "another reason".
- Different F' provide different explanations.

So we have to "explain" unsatisfiability for $F \in \mathcal{MU}$.

Understanding UNSAT — understanding MU

For unsatisfiable $F \in \mathcal{USAT}$

we want to **"understand"** its unsatisfiability.

We want to SEE it.

Each $F' \subseteq F$ with $F' \in \mathcal{MU}$ contains one explanation:

- The additional clauses in $F \setminus F'$ can make the contradiction of F' more easily accessible, but do not contribute "another reason".
- Different F' provide different explanations.

So we have to "explain" unsatisfiability for $F \in \mathcal{MU}$.

Explain $\mathcal{MU}'(k)$ for $k = 1, 2, 3, \ldots$.

The Classification Conjecture

Conjecture *For every $k \in \mathbb{N}$ there are finitely many "patterns", which explain $\mathcal{MU}'(k)$ completely.*

1. $\mathcal{MU}'(1) = \{\perp\}$
2. $\mathcal{MU}'(2) =$ cycles of length $n \geq 2$ plus "at least one variable is false" plus "at least one variable is true".
3. $\mathcal{MU}'(3)$: in preparation.

Two problems

For $F \in \mathcal{MU}$ let

$$\mathbf{sDP(F)} := \{F' \in \mathcal{MU}' : F \xrightarrow{\text{sDP}}_* F'\}.$$

First problem: We want to understand $F \in \mathcal{MU}(k)$. For that we consider $\text{sDP}(F)$. Note $\text{sDP}(F) \subset \mathcal{MU}(k)$.

Easiest is $|\text{sDP}(F)| = 1$ — when does this hold?

Second problem: A transition $F \xrightarrow{\text{sDP}}_* F' \in \mathcal{MU}'$ removes "trivialities" — what if we have to understand these trivialities?

The "singularity index"

Theorem

For $F \in \mathcal{MU}$ and $F', F'' \in \mathcal{MU}$ we have $n(F') = n(F'')$.

This allows us to define the **singularity index si(F)** $\in \mathbb{N}_0$ as $\mathrm{si}(F) := n(F) - n(F')$ for some $F' \in \mathrm{sDP}(F)$.

Corollary

If $F \in \mathcal{MU}(2)$, then for $F', F'' \in \mathrm{sDP}(F)$ we have $F' \cong F''$.

Here $F' \cong F''$ denotes isomorphism.

Confluence

Theorem

If $F \in \mathcal{SMU}$, then $|\mathrm{sDP}(F)| = 1$.

Confluence modulo isomorphism

Theorem

If for $F \in \mathcal{MU}$ we have $\mathrm{sDP}(F) \subseteq \mathcal{SMU}$, then for $F', F'' \in \mathrm{sDP}(F)$ we have $F' \cong F''$.

Corollary

If $F \in \mathcal{MU}(2)$, then for $F', F'' \in \mathrm{sDP}(F)$ we have $F' \cong F''$.

Commutativity

In Kullmann and Luckhardt [7, 8] is it shown:

Lemma

For $F \in \mathcal{CLS}$, variables $v_1, \dots, v_k$ and a permutation $\pi \in S_k$ we have that

$$\mathrm{DP}_{v_1,\dots,v_k}(F), \quad \mathrm{DP}_{v_{\pi(1)},\dots,v_{\pi(k)}}(F)$$

are equal modulo subsumption.

Controlled permutations

- The key is to exchange neighbouring sDP-reductions such that we again obtain an sDP-reduction.
- Distinguish between 1-singular variables (occurring in both signs only once) and non-1-singular variables.

Conclusions

Contributions:

- For minimally unsatisfiable clause-sets the number of sDP-reductions until reaching nonsingularity does not depend on the choice of reductions.
- Confluence of sDP-reduction for saturated minimally unsatisfiable clause-sets.
- Confluence modulo isomorphism of sDP-reduction in case all maximal sDP-reductions yield saturated minimally unsatisfiable clause-sets.
- Obtaining unique normalforms (up to isomorphism) for $\mathcal{MU}(2)$ via sDP-reduction.

Next steps:

- Fuller understanding of sDP-reduction.
- Generalisation to all of $\mathcal{CLS}$.

$\mathfrak{End}$

(references on the remaining slides).

Bibliography I

[1] Ron Aharoni and Nathan Linial. Minimal non-two-colorable hypergraphs and minimal unsatisfiable formulas. *Journal of Combinatorial Theory*, A 43:196–204, 1986.

[2] Martin Davis and Hilary Putnam. A computing procedure for quantification theory. *Journal of the ACM*, 7:201–215, 1960.

[3] Gennady Davydov, Inna Davydova, and Hans Kleine Büning. An efficient algorithm for the minimal unsatisfiability problem for a subclass of CNF. *Annals of Mathematics and Artificial Intelligence*, 23:229–245, 1998.

[4] Hans Kleine Büning. On subclasses of minimal unsatisfiable formulas. *Discrete Applied Mathematics*, 107:83–98, 2000.

Bibliography II

[5] Hans Kleine Büning and Oliver Kullmann. Minimal unsatisfiability and autarkies. In Armin Biere, Marijn J.H. Heule, Hans van Maaren, and Toby Walsh, editors, *Handbook of Satisfiability*, volume 185 of *Frontiers in Artificial Intelligence and Applications*, chapter 11, pages 339–401. IOS Press, February 2009. ISBN 978-1-58603-929-5. doi: 10.3233/978-1-58603-929-5-339.

[6] Oliver Kullmann. An application of matroid theory to the SAT problem. In *Fifteenth Annual IEEE Conference on Computational Complexity*, pages 116–124. IEEE Computer Society, July 2000. See also TR00-018, Electronic Colloquium on Computational Complexity (ECCC), March 2000.

Bibliography III

[7] Oliver Kullmann and Horst Luckhardt. Deciding propositional tautologies: Algorithms and their complexity. Preprint, 82 pages; the ps-file can be obtained at http://cs.swan.ac.uk/~csoliver/Artikel/tg.ps, January 1997.

[8] Oliver Kullmann and Horst Luckhardt. Algorithms for SAT/TAUT decision based on various measures. Preprint, 71 pages; the ps-file can be obtained from http://cs.swan.ac.uk/~csoliver/Artikel/TAUT.ps, February 1999.

[9] Oliver Kullmann and Xishun Zhao. On Davis-Putnam reductions for minimally unsatisfiable clause-sets. In Alessandro Cimatti and Roberto Sebastiani, editors, *Theory and Applications of Satisfiability Testing - SAT 2012*, volume LNCS 7317 of *Lecture Notes in Computer Science*, pages 270–283. Springer, 2012.

Bibliography IV

[10] Oliver Kullmann and Xishun Zhao. On Davis-Putnam reductions for minimally unsatisfiable clause-sets. Technical Report arXiv:1202.2600v3 [cs.DM], arXiv, May 2012.