On Efficient Computation of Variable MUSes

Anton Belov[1], Alexander Ivrii[2],
Arie Matsliah[2], Joao Marques-Silva[1]

[1]Complex and Adaptive Systems Laboratory
University College Dublin, Ireland

[2]IBM Research – Haifa, Israel

SAT 2012
June 20, 2012
Trento, Italy

Introduction

$\mathcal{F} = \{C_1, \ldots, C_6\} \in \text{UNSAT}$

$C_1 = (p)$ $\quad$ $C_3 = (\neg p \vee \neg q)$ $\quad$ $C_5 = (\neg p \vee r)$

$C_2 = (q)$ $\quad$ $C_4 = (p \vee q)$ $\quad$ $C_6 = (\neg q \vee \neg r)$

Introduction

$\mathcal{F} = \{C_1, \ldots, C_6\} \in \text{UNSAT}$

$C_1 = (p)$ $\quad$ $C_3 = (\neg p \vee \neg q)$ $\quad$ $C_5 = (\neg p \vee r)$

$C_2 = (q)$ $\quad$ $C_4 = (p \vee q)$ $\quad$ $C_6 = (\neg q \vee \neg r)$

Which part of $\mathcal{F}$ is responsible for its inconsistency ?

Introduction

$\mathcal{F} = \{C_1, \ldots, C_6\} \in \text{UNSAT}$

$C_1 = (p)$	$C_3 = (\neg p \vee \neg q)$	$C_5 = (\neg p \vee r)$
$C_2 = (q)$	$C_4 = (p \vee q)$	$C_6 = (\neg q \vee \neg r)$

Which part of $\mathcal{F}$ is responsible for its inconsistency ?

Introduction

$\mathcal{F} = \{C_1, \ldots, C_6\} \in$ UNSAT

$C_1 = (p)$	$C_3 = (\neg p \vee \neg q)$	$C_5 = (\neg p \vee r)$
$C_2 = (q)$	$C_4 = (p \vee q)$	$C_6 = (\neg q \vee \neg r)$

Which part of $\mathcal{F}$ is responsible for its inconsistency ?

$\{C_1, C_2, C_3\}$ is a subset-minimal set of clauses required to refute $\mathcal{F}$.

Introduction

$\mathcal{F} = \{C_1, \ldots, C_6\} \in$ UNSAT

$C_1 = (p)$	$C_3 = (\neg p \vee \neg q)$	$C_5 = (\neg p \vee r)$
$C_2 = (q)$	$C_4 = (p \vee q)$	$C_6 = (\neg q \vee \neg r)$

Which part of $\mathcal{F}$ is responsible for its inconsistency ?

$\{C_1, C_2, C_3\}$ is a subset-minimal set of clauses required to refute $\mathcal{F}$.

$\mathcal{F}' \subseteq \mathcal{F}$ is *minimally unsatisfiable subformula (MUS)* of $\mathcal{F}$ if $\mathcal{F}' \in$ UNSAT, and $\forall C \in \mathcal{F}', \mathcal{F}' \setminus \{C\} \in$ SAT.

Introduction

$\mathcal{F} = \{C_1, \ldots, C_6\} \in \text{UNSAT}$

$C_1 = (p)$

$C_2 = (q)$

$C_3 = (\neg p \vee \neg q)$

$C_4 = (p \vee q)$

$C_5 = (\neg p \vee r)$

$C_6 = (\neg q \vee \neg r)$

Which part of $\mathcal{F}$ is responsible for its inconsistency ?

$\{C_1, C_2, C_3\}$ is a subset-minimal set of clauses required to refute $\mathcal{F}$.

$\mathcal{F}' \subseteq \mathcal{F}$ is *minimally unsatisfiable subformula (MUS)* of $\mathcal{F}$ if $\mathcal{F}' \in \text{UNSAT}$, and $\forall C \in \mathcal{F}', \mathcal{F}' \setminus \{C\} \in \text{SAT}$.

$\{C_1, C_2, C_5, C_6\}$ is also an MUS of $\mathcal{F}$.

Introduction

$\mathcal{F} = \mathcal{G}_1 \cup \mathcal{G}_2 \cup \mathcal{G}_3 \in$ UNSAT — partitioned into *groups* (sets) of clauses

$$\mathcal{G}_1 = \{C_1, C_2\}, \quad \mathcal{G}_2 = \{C_3, C_4\}, \quad \mathcal{G}_3 = \{C_5, C_6\}.$$

$C_1 = (p)$	$C_3 = (\neg p \vee \neg q)$	$C_5 = (\neg p \vee r)$
$C_2 = (q)$	$C_4 = (p \vee q)$	$C_6 = (\neg q \vee \neg r)$

Introduction

$\mathcal{F} = \mathcal{G}_1 \cup \mathcal{G}_2 \cup \mathcal{G}_3 \in$ UNSAT — partitioned into *groups* (sets) of clauses

$$\mathcal{G}_1 = \{C_1, C_2\}, \quad \mathcal{G}_2 = \{C_3, C_4\}, \quad \mathcal{G}_3 = \{C_5, C_6\}.$$

$C_1 = (p)$	$C_3 = (\neg p \vee \neg q)$	$C_5 = (\neg p \vee r)$
$C_2 = (q)$	$C_4 = (p \vee q)$	$C_6 = (\neg q \vee \neg r)$

What is a subset-minimal set of *groups* required to refute $\mathcal{F}$?

Introduction

$\mathcal{F} = \mathcal{G}_1 \cup \mathcal{G}_2 \cup \mathcal{G}_3 \in$ UNSAT — partitioned into *groups* (sets) of clauses

$$\mathcal{G}_1 = \{C_1, C_2\}, \quad \mathcal{G}_2 = \{C_3, C_4\}, \quad \mathcal{G}_3 = \{C_5, C_6\}.$$

$C_1 = (p)$	$C_3 = (\neg p \vee \neg q)$	$C_5 = (\neg p \vee r)$
$C_2 = (q)$	$C_4 = (p \vee q)$	$C_6 = (\neg q \vee \neg r)$

What is a subset-minimal set of *groups* required to refute $\mathcal{F}$?

$\{\mathcal{G}_1, \mathcal{G}_2\}$ is a *group-MUS* of (the partitioned) $\mathcal{F}$.

Introduction

$\mathcal{F} = \mathcal{G}_1 \cup \mathcal{G}_2 \cup \mathcal{G}_3 \in$ UNSAT — partitioned into *groups* (sets) of clauses

$$\mathcal{G}_1 = \{C_1, C_2\}, \quad \mathcal{G}_2 = \{C_3, C_4\}, \quad \mathcal{G}_3 = \{C_5, C_6\}.$$

$C_1 = (p)$	$C_3 = (\neg p \vee \neg q)$	$C_5 = (\neg p \vee r)$
$C_2 = (q)$	$C_4 = (p \vee q)$	$C_6 = (\neg q \vee \neg r)$

What is a subset-minimal set of *groups* required to refute $\mathcal{F}$?

$\{\mathcal{G}_1, \mathcal{G}_2\}$ is a *group-MUS* of (the partitioned) $\mathcal{F}$.

$\{\mathcal{G}_1, \mathcal{G}_3\}$ is also a *group-MUS* of (the partitioned) $\mathcal{F}$.

Introduction

What about the variables of $\mathcal{F}$?

$$C_1 = (p) \qquad C_3 = (\neg p \vee \neg q) \qquad C_5 = (\neg p \vee r)$$

$$C_2 = (q) \qquad C_4 = (p \vee q) \qquad C_6 = (\neg q \vee \neg r)$$

What is a subset-minimal set of variables required to refute $\mathcal{F}$?

Introduction

What about the variables of $\mathcal{F}$?

$C_1 = (p)$	$C_3 = (\neg p \vee \neg q)$	$C_5 = (\neg p \vee r)$
$C_2 = (q)$	$C_4 = (p \vee q)$	$C_6 = (\neg q \vee \neg r)$

What is a subset-minimal set of variables required to refute $\mathcal{F}$?

$\{p, q\}$ is such a subset of variables.

Introduction

What about the variables of $\mathcal{F}$?

$C_1 = (p)$	$C_3 = (\neg p \vee \neg q)$	$C_5 = (\neg p \vee r)$
$C_2 = (q)$	$C_4 = (p \vee q)$	$C_6 = (\neg q \vee \neg r)$

What is a subset-minimal set of variables required to refute $\mathcal{F}$?

$\{p, q\}$ is such a subset of variables.

$\{p, q\}$ is a *variable-MUS* of $\mathcal{F}$.

Introduction

What about the variables of $\mathcal{F}$?

$C_1 = (p)$ $C_3 = (\neg p \vee \neg q)$ $C_5 = (\neg p \vee r)$

$C_2 = (q)$ $C_4 = (p \vee q)$ $C_6 = (\neg q \vee \neg r)$

What is a subset-minimal set of variables required to refute $\mathcal{F}$?

$\{p, q\}$ is such a subset of variables.

$\{p, q\}$ is a *variable-MUS* of $\mathcal{F}$.

This paper is about algorithms for efficient computation of variable-MUSes.

Introduction

What about the variables of $\mathcal{F}$?

$C_1 = (p)$ $C_3 = (\neg p \vee \neg q)$ $C_5 = (\neg p \vee r)$

$C_2 = (q)$ $C_4 = (p \vee q)$ $C_6 = (\neg q \vee \neg r)$

What is a subset-minimal set of variables required to refute $\mathcal{F}$?

$\{p, q\}$ is such a subset of variables.

$\{p, q\}$ is a *variable-MUS* of $\mathcal{F}$.

This paper is about algorithms for efficient computation of variable-MUSes.

Some applications: finding vertex-critical subgraphs; abstraction in abstraction refinement framework; satisfying assignments minimization.

Variable-MUSes [Zhen-Yu Chen and De-Cheng Ding, TAMC'06]

Definition

Let $V \subseteq Var(\mathcal{F})$. The subformula of $\mathcal{F}$ *induced* by V is the formula $\mathcal{F}|_V = \{C \mid C \in \mathcal{F} \text{ and } Var(C) \subseteq V\}$.

I.e. $\mathcal{F}|_V$ includes only those clauses of $\mathcal{F}$ whose variables are in V.

Variable-MUSes [Zhen-Yu Chen and De-Cheng Ding, TAMC'06]

Definition

Let $V \subseteq Var(\mathcal{F})$. The subformula of $\mathcal{F}$ *induced* by V is the formula $\mathcal{F}|_V = \{C \mid C \in \mathcal{F} \text{ and } Var(C) \subseteq V\}$.

I.e. $\mathcal{F}|_V$ includes only those clauses of $\mathcal{F}$ whose variables are in V.

$$C_1 = (p) \qquad C_3 = (\neg p \vee \neg q) \qquad C_5 = (\neg p \vee r)$$

$$C_2 = (q) \qquad C_4 = (p \vee q) \qquad C_6 = (\neg q \vee \neg r)$$

Variable-MUSes [Zhen-Yu Chen and De-Cheng Ding, TAMC'06]

Definition

Let $V \subseteq Var(\mathcal{F})$. The subformula of $\mathcal{F}$ *induced* by V is the formula $\mathcal{F}|_V = \{C \mid C \in \mathcal{F} \text{ and } Var(C) \subseteq V\}$.

I.e. $\mathcal{F}|_V$ includes only those clauses of $\mathcal{F}$ whose variables are in V.

$C_1 = (p)$ $\quad$ $C_3 = (\neg p \vee \neg q)$ $\quad$ $C_5 = (\neg p \vee r)$

$C_2 = (q)$ $\quad$ $C_4 = (p \vee q)$ $\quad$ $C_6 = (\neg q \vee \neg r)$

- The subformula induced by $\{p, q\}$ is $\mathcal{F}|_{\{p,q\}} = \{C_1, C_2, C_3, C_4\}$.
  - variable r is "removed" from $\mathcal{F}$.

Variable-MUSes [Zhen-Yu Chen and De-Cheng Ding, TAMC'06]

Definition

Let $V \subseteq Var(\mathcal{F})$. The subformula of $\mathcal{F}$ *induced* by V is the formula $\mathcal{F}|_V = \{C \mid C \in \mathcal{F} \text{ and } Var(C) \subseteq V\}$.

I.e. $\mathcal{F}|_V$ includes only those clauses of $\mathcal{F}$ whose variables are in V.

$C_1 = (p)$ $C_3 = (\neg p \vee \neg q)$ $C_5 = (\neg p \vee r)$

$C_2 = (q)$ $C_4 = (p \vee q)$ $C_6 = (\neg q \vee \neg r)$

- The subformula induced by $\{p, q\}$ is $\mathcal{F}|_{\{p,q\}} = \{C_1, C_2, C_3, C_4\}$.
  - variable r is "removed" from $\mathcal{F}$.
- The subformula induced by $\{p\}$, $\mathcal{F}|_{\{p\}} = \{C_1\}$.
  - variables q, r are "removed" from $\mathcal{F}$.

Variable-MUSes [Zhen-Yu Chen and De-Cheng Ding, TAMC'06]

Definition

Let $V \subseteq Var(\mathcal{F})$. The subformula of $\mathcal{F}$ *induced* by V is the formula $\mathcal{F}|_V = \{C \mid C \in \mathcal{F} \text{ and } Var(C) \subseteq V\}$.

I.e. $\mathcal{F}|_V$ includes only those clauses of $\mathcal{F}$ whose variables are in V.

Definition

A set $V \subset Var(\mathcal{F})$ is a *variable-MUS (VMUS)* of $\mathcal{F}$ if $\mathcal{F}|_V \in$ UNSAT, and for any $V' \subset V$, $\mathcal{F}|_{V'} \in$ SAT.

Variable-MUSes [Zhen-Yu Chen and De-Cheng Ding, TAMC'06]

Definition

Let $V \subseteq Var(\mathcal{F})$. The subformula of $\mathcal{F}$ *induced* by V is the formula $\mathcal{F}|_V = \{C \mid C \in \mathcal{F} \text{ and } Var(C) \subseteq V\}$.

I.e. $\mathcal{F}|_V$ includes only those clauses of $\mathcal{F}$ whose variables are in V.

Definition

A set $V \subset Var(\mathcal{F})$ is a *variable-MUS (VMUS)* of $\mathcal{F}$ if $\mathcal{F}|_V \in$ UNSAT, and for any $V' \subset V$, $\mathcal{F}|_{V'} \in$ SAT.

$$C_1 = (p) \qquad C_3 = (\neg p \vee \neg q) \qquad C_5 = (\neg p \vee r)$$

$$C_2 = (q) \qquad C_4 = (p \vee q) \qquad C_6 = (\neg q \vee \neg r)$$

Variable-MUSes [Zhen-Yu Chen and De-Cheng Ding, TAMC'06]

Definition

Let $V \subseteq Var(\mathcal{F})$. The subformula of $\mathcal{F}$ *induced* by V is the formula $\mathcal{F}|_V = \{C \mid C \in \mathcal{F} \text{ and } Var(C) \subseteq V\}$.

I.e. $\mathcal{F}|_V$ includes only those clauses of $\mathcal{F}$ whose variables are in V.

Definition

A set $V \subset Var(\mathcal{F})$ is a *variable-MUS (VMUS)* of $\mathcal{F}$ if $\mathcal{F}|_V \in$ UNSAT, and for any $V' \subset V$, $\mathcal{F}|_{V'} \in$ SAT.

$$C_1 = (p) \qquad C_3 = (\neg p \vee \neg q) \qquad C_5 = (\neg p \vee r)$$

$$C_2 = (q) \qquad C_4 = (p \vee q) \qquad C_6 = (\neg q \vee \neg r)$$

- $\mathcal{F}|_{\{p,q\}} = \{C_1, C_2, C_3, C_4\} \in$ UNSAT.

Variable-MUSes [Zhen-Yu Chen and De-Cheng Ding, TAMC'06]

Definition

Let $V \subseteq Var(\mathcal{F})$. The subformula of $\mathcal{F}$ *induced* by V is the formula $\mathcal{F}|_V = \{C \mid C \in \mathcal{F} \text{ and } Var(C) \subseteq V\}$.

I.e. $\mathcal{F}|_V$ includes only those clauses of $\mathcal{F}$ whose variables are in V.

Definition

A set $V \subset Var(\mathcal{F})$ is a *variable-MUS (VMUS)* of $\mathcal{F}$ if $\mathcal{F}|_V \in$ UNSAT, and for any $V' \subset V$, $\mathcal{F}|_{V'} \in$ SAT.

$C_1 = (p)$ $C_3 = (\neg p \vee \neg q)$ $C_5 = (\neg p \vee r)$

$C_2 = (q)$ $C_4 = (p \vee q)$ $C_6 = (\neg q \vee \neg r)$

- $\mathcal{F}|_{\{p\}} = \{C_1\} \in$ SAT.

Variable-MUSes [Zhen-Yu Chen and De-Cheng Ding, TAMC'06]

Definition

Let $V \subseteq Var(\mathcal{F})$. The subformula of $\mathcal{F}$ *induced* by V is the formula $\mathcal{F}|_V = \{C \mid C \in \mathcal{F} \text{ and } Var(C) \subseteq V\}$.

I.e. $\mathcal{F}|_V$ includes only those clauses of $\mathcal{F}$ whose variables are in V.

Definition

A set $V \subset Var(\mathcal{F})$ is a *variable-MUS (VMUS)* of $\mathcal{F}$ if $\mathcal{F}|_V \in$ UNSAT, and for any $V' \subset V$, $\mathcal{F}|_{V'} \in$ SAT.

$C_1 = (p)$ $C_3 = (\neg p \vee \neg q)$ $C_5 = (\neg p \vee r)$

$C_2 = (q)$ $C_4 = (p \vee q)$ $C_6 = (\neg q \vee \neg r)$

- $\mathcal{F}|_{\{q\}} = \{C_2\} \in$ SAT.

Variable-MUSes [Zhen-Yu Chen and De-Cheng Ding, TAMC'06]

Definition

Let $V \subseteq Var(\mathcal{F})$. The subformula of $\mathcal{F}$ *induced* by V is the formula $\mathcal{F}|_V = \{C \mid C \in \mathcal{F} \text{ and } Var(C) \subseteq V\}$.

I.e. $\mathcal{F}|_V$ includes only those clauses of $\mathcal{F}$ whose variables are in V.

Definition

A set $V \subset Var(\mathcal{F})$ is a *variable-MUS (VMUS)* of $\mathcal{F}$ if $\mathcal{F}|_V \in$ UNSAT, and for any $V' \subset V$, $\mathcal{F}|_{V'} \in$ SAT.

$$C_1 = (p) \qquad C_3 = (\neg p \vee \neg q) \qquad C_5 = (\neg p \vee r)$$

$$C_2 = (q) \qquad C_4 = (p \vee q) \qquad C_6 = (\neg q \vee \neg r)$$

- Hence, $\{p, q\}$ is a VMUS of $\mathcal{F}$. Notation: $\{p, q\} \in \text{VMUS}(\mathcal{F})$.

Computing VMUSes

Basic algorithms are similar to MUS extraction algorithms: based on detection of *necessary* variables.

Notation: for $v \in Var(\mathcal{F})$, $\mathcal{F}^v = \{C \mid C \in \mathcal{F} \text{ and } v \in Var(C)\}$.

Definition

$v \in Var(\mathcal{F})$ is *necessary* for $\mathcal{F}$ if $\mathcal{F} \in$ UNSAT and $\mathcal{F} \setminus \mathcal{F}^v \in$ SAT.

Computing VMUSes

Basic algorithms are similar to MUS extraction algorithms: based on detection of *necessary* variables.

Notation: for $v \in Var(\mathcal{F})$, $\mathcal{F}^v = \{C \mid C \in \mathcal{F} \text{ and } v \in Var(C)\}$.

Definition

$v \in Var(\mathcal{F})$ is *necessary* for $\mathcal{F}$ if $\mathcal{F} \in$ UNSAT and $\mathcal{F} \setminus \mathcal{F}^v \in$ SAT.

Properties:

1. $V \in \text{VMUS}(\mathcal{F})$ if and only if every $v \in V$ is necessary for $\mathcal{F}|_V$.
2. If v is necessary for $\mathcal{F}$, then v is necessary for any unsatisfiable $\mathcal{F}' \subseteq \mathcal{F}$.

Hybrid VMUS Computation w/o optimizations

Input $\mapsto$ **Output**: Unsatisfiable CNF Formula $\mathcal{F}$ $\mapsto$ $V \in \mathrm{VMUS}(\mathcal{F})$

$V \leftarrow \emptyset$ // VMUS under-approximation

$V_w \leftarrow Var(\mathcal{F})$ // Working (‘‘untested’’) set of variables

$\mathcal{F}_w \leftarrow \mathcal{F}$ // Working formula

while $V_w \neq \emptyset$ **do** // Inv: $\mathcal{F}_w = \mathcal{F}|_{V \cup V_w}$ and $\forall v \in V$ is nec. for $\mathcal{F}_w$

- $v \leftarrow \mathtt{PickVariable}(V_w)$
- $V_w \leftarrow V_w \setminus \{v\}$
- $\mathcal{R} \leftarrow \mathtt{CNF}(\neg \mathcal{F}_w^v)$ // Redundancy removal
- $\mathrm{st} = \mathtt{SAT}(\mathcal{F}_w \setminus \mathcal{F}_w^v)$
- **if** st $=$ false **then** // v is not necessary for $\mathcal{F}_w$
  - $V_w \leftarrow V_w \cap Var(\mathcal{U})$ // Variable-set refinement
  - $\mathcal{F}_w \leftarrow \mathcal{F}_w \setminus \mathcal{F}_w^v$
- **else** // v is necessary for $\mathcal{F}_w$
  - $V \leftarrow V \cup \{v\}$
  - $V_w \leftarrow V_w \setminus V$

return V // $V \in \mathrm{VMUS}(\mathcal{F})$ and $\mathcal{F}_w = \mathcal{F}|_V \in \mathrm{VMU}$

Hybrid VMUS Computation w/o optimizations

Similar to the `Removal` algorithm [C. Desrosiers, et al, J. Comb. Optim. 2008]

Does not scale:

- Number of SAT solver calls: $|Var(\mathcal{F})|$.
- 295 benchmarks from SAT Comp 2011, TO = 1800 sec, MO = 4GB.
  - W/o optimizations solved: 157, avg.vars: 6198, max.vars: 49490.

Hybrid VMUS Computation w/o optimizations

Similar to the `Removal` algorithm [C. Desrosiers, et al, J. Comb. Optim. 2008]

Does not scale:

- Number of SAT solver calls: $|Var(\mathcal{F})|$.
- 295 benchmarks from SAT Comp 2011, TO = 1800 sec, MO = 4GB.
  - W/o optimizations solved: 157, avg.vars: 6198, max.vars: 49490.
  - With optimizations solved: 265, avg.vars: 105044; max.vars: 1198007.

Hybrid VMUS Computation w/o optimizations

Similar to the `Removal` algorithm [C. Desrosiers, et al, J. Comb. Optim. 2008]

Does not scale:

- Number of SAT solver calls: $|Var(\mathcal{F})|$.
- 295 benchmarks from SAT Comp 2011, TO = 1800 sec, MO = 4GB.
  - W/o optimizations solved: 157, avg.vars: 6198, max.vars: 49490.
  - With optimizations solved: 265, avg.vars: 105044; max.vars: 1198007.
- Bottleneck: SAT solver calls.

Hybrid VMUS Computation w/o optimizations

Similar to the `Removal` algorithm [C. Desrosiers, et al, J. Comb. Optim. 2008]

Does not scale:

- Number of SAT solver calls: $|Var(\mathcal{F})|$.
- 295 benchmarks from SAT Comp 2011, TO = 1800 sec, MO = 4GB.
  - W/o optimizations solved: 157, avg.vars: 6198, max.vars: 49490.
  - With optimizations solved: 265, avg.vars: 105044; max.vars: 1198007.
- Bottleneck: SAT solver calls.

Optimizations are aimed at:

- Making fewer SAT solver calls.
- Making SAT solver calls easier.

Optimizations: variable-set refinement

- **Fact:** Let $\mathcal{U}$ be an unsatisfiable core of $\mathcal{F}$. Then, $Var(\mathcal{U})$ contains at least one VMUS of $\mathcal{F}$.

Optimizations: variable-set refinement

- **Fact:** Let $\mathcal{U}$ be an unsatisfiable core of $\mathcal{F}$. Then, $Var(\mathcal{U})$ contains at least one VMUS of $\mathcal{F}$.
- Hence, if $\mathcal{U}$ is an unsatisfiable core of $\mathcal{F}$, all variables outside of $Var(\mathcal{U})$ can be removed from $\mathcal{F}$ — *variable-set refinement*.

Optimizations: variable-set refinement

- **Fact:** Let $\mathcal{U}$ be an unsatisfiable core of $\mathcal{F}$. Then, $Var(\mathcal{U})$ contains at least one VMUS of $\mathcal{F}$.
- Hence, if $\mathcal{U}$ is an unsatisfiable core of $\mathcal{F}$, all variables outside of $Var(\mathcal{U})$ can be removed from $\mathcal{F}$ — *variable-set refinement*.
- Relies on the capability of SAT solvers to return unsatisfiable core.

Optimizations: variable-set refinement

- **Fact:** Let $\mathcal{U}$ be an unsatisfiable core of $\mathcal{F}$. Then, $Var(\mathcal{U})$ contains at least one VMUS of $\mathcal{F}$.
- Hence, if $\mathcal{U}$ is an unsatisfiable core of $\mathcal{F}$, all variables outside of $Var(\mathcal{U})$ can be removed from $\mathcal{F}$ — *variable-set refinement*.
- Relies on the capability of SAT solvers to return unsatisfiable core.
- Applied to the working set of variables V_w inside the main loop, $\mathcal{F}_w$ is updated accordingly.

Optimizations: variable-set refinement

- **Fact:** Let $\mathcal{U}$ be an unsatisfiable core of $\mathcal{F}$. Then, $Var(\mathcal{U})$ contains at least one VMUS of $\mathcal{F}$.
- Hence, if $\mathcal{U}$ is an unsatisfiable core of $\mathcal{F}$, all variables outside of $Var(\mathcal{U})$ can be removed from $\mathcal{F}$ — *variable-set refinement*.
- Relies on the capability of SAT solvers to return unsatisfiable core.
- Applied to the working set of variables V_w inside the main loop, $\mathcal{F}_w$ is updated accordingly.
- Effect: remove multiple unnecessary variables in one SAT solver call.

Hybrid VMUS Computation: variable-set refinement

Input $\mapsto$ **Output**: Unsatisfiable CNF Formula $\mathcal{F}$ $\mapsto$ $V \in \mathrm{VMUS}(\mathcal{F})$

$V \leftarrow \emptyset$ // VMUS under-approximation

$V_w \leftarrow Var(\mathcal{F})$ // Working (''untested'') set of variables

$\mathcal{F}_w \leftarrow \mathcal{F}$ // Working formula

while $V_w \neq \emptyset$ **do** // Inv: $\mathcal{F}_w = \mathcal{F}|_{V \cup V_w}$ and $\forall v \in V$ is nec. for $\mathcal{F}_w$

- $v \leftarrow \mathtt{PickVariable}(V_w)$
- $V_w \leftarrow V_w \setminus \{v\}$
- $\mathcal{R} \leftarrow \mathtt{CNF}(\neg \mathcal{F}_w^v)$ // Redundancy removal
- $\mathrm{st} = \mathtt{SAT}(\mathcal{F}_w \setminus \mathcal{F}_w^v)$
- **if** st $=$ false **then** // v is not necessary for $\mathcal{F}_w$
  - $V_w \leftarrow V_w \cap Var(\mathcal{U})$ // Variable-set refinement
  - $\mathcal{F}_w \leftarrow \mathcal{F}_w \setminus \mathcal{F}_w^v$
- **else** // v is necessary for $\mathcal{F}_w$
  - $V \leftarrow V \cup \{v\}$
  - $V_w \leftarrow V_w \setminus V$

return V // $V \in \mathrm{VMUS}(\mathcal{F})$ and $\mathcal{F}_w = \mathcal{F}|_V \in \mathrm{VMU}$

Hybrid VMUS Computation: variable-set refinement

Input $\mapsto$ **Output**: Unsatisfiable CNF Formula $\mathcal{F}$ $\mapsto$ $V \in \mathrm{VMUS}(\mathcal{F})$

$V \leftarrow \emptyset$ // VMUS under-approximation

$V_w \leftarrow Var(\mathcal{F})$ // Working (‘‘untested’’) set of variables

$\mathcal{F}_w \leftarrow \mathcal{F}$ // Working formula

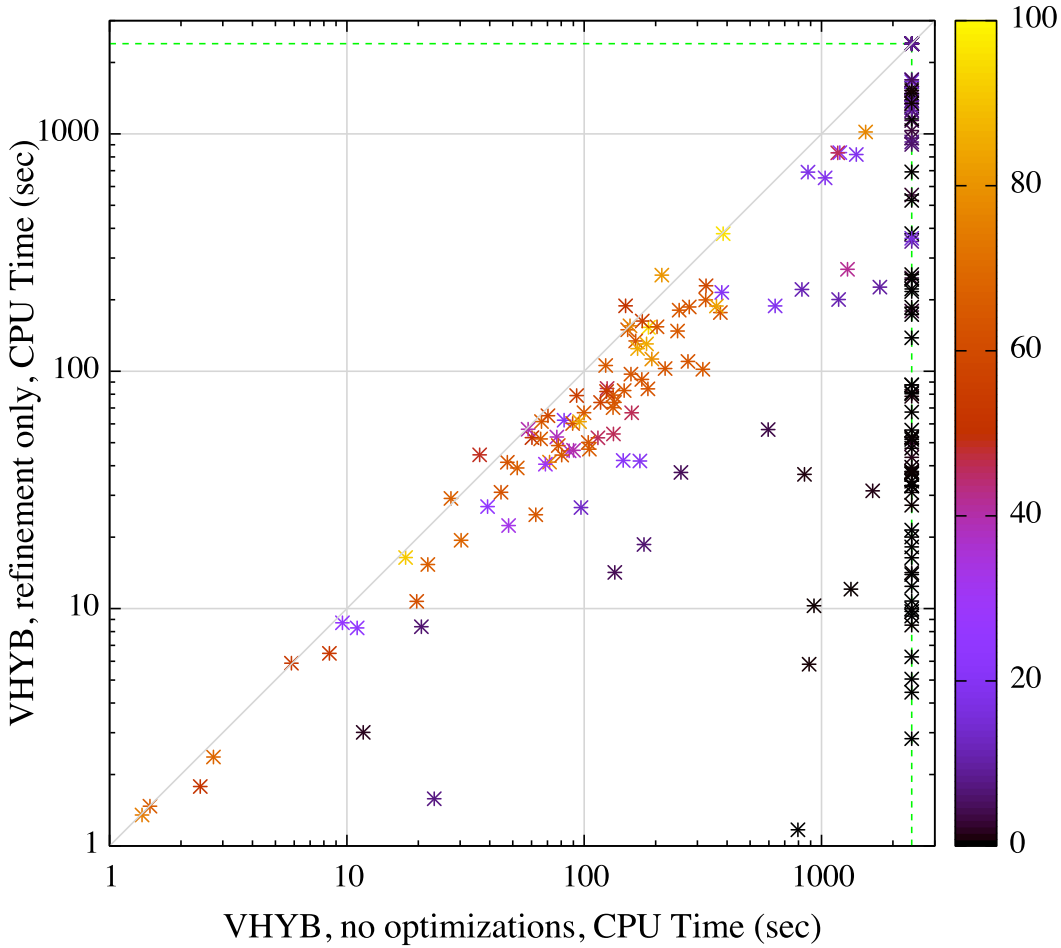
while $V_w \neq \emptyset$ **do** // Inv: $\mathcal{F}_w = \mathcal{F}|_{V \cup V_w}$ and $\forall v \in V$ is nec. for $\mathcal{F}_w$

- $v \leftarrow \mathtt{PickVariable}(V_w)$
- $V_w \leftarrow V_w \setminus \{v\}$
- $\mathcal{R} \leftarrow \mathtt{CNF}(\neg \mathcal{F}_w^v)$ // Redundancy removal
- $\langle \mathrm{st}, \mathcal{U} \rangle = \mathtt{SAT}(\mathcal{F}_w \setminus \mathcal{F}_w^v)$
- **if** st = false **then** // v is not necessary for $\mathcal{F}_w$
  - $V_w \leftarrow V_w \cap Var(\mathcal{U})$ // Variable-set refinement
  - $\mathcal{F}_w \leftarrow \mathcal{F}_w|_{V \cup V_w}$
- **else** // v is necessary for $\mathcal{F}_w$
  - $V \leftarrow V \cup \{v\}$
  - $V_w \leftarrow V_w \setminus V$

return V // $V \in \mathrm{VMUS}(\mathcal{F})$ and $\mathcal{F}_w = \mathcal{F}|_V \in \mathrm{VMU}$

Impact of variable-set refinement

- 295 benchmarks from SAT Comp 2011, TO = 1800 sec, MO = 4GB.



- CPU Time, w/o optimizations (#sol=157) vs refinement (#sol=239)
- Color: VMUS size (% of the number of variables in the input).

Optimizations: variable-based model rotation (VMR)

- **Fact:** Take $\mathcal{F} \in$ UNSAT and an assignment τ to $Var(\mathcal{F})$. Then, any variable shared among the clauses falsified by τ is necessary for $\mathcal{F}$.

Optimizations: variable-based model rotation (VMR)

- **Fact:** Take $\mathcal{F} \in$ UNSAT and an assignment τ to $Var(\mathcal{F})$. Then, any variable shared among the clauses falsified by τ is necessary for $\mathcal{F}$.
- Why ?
  - Let $\mathcal{F}'$ be the clauses of $\mathcal{F}$ falsified by τ.
  - Let v be any variable shared among the clauses of $\mathcal{F}'$
    - i.e. $\forall C \in \mathcal{F}'$, $v \in Var(C)$.
  - Remove v from $\mathcal{F}$.
  - All clauses of $\mathcal{F}'$ will be gone, because they all contain v.
  - We get a subformula of $\mathcal{F} \setminus \mathcal{F}'$.
  - But, $\mathcal{F} \setminus \mathcal{F}' \in \mathtt{SAT}$, since τ is its model.
  - Hence, $\mathcal{F} \setminus \mathcal{F}^v \in \mathtt{SAT}$, and so v is necessary for $\mathcal{F}$.

Optimizations: variable-based model rotation (VMR)

- **Fact:** Take $\mathcal{F} \in$ UNSAT and an assignment τ to $Var(\mathcal{F})$. Then, any variable shared among the clauses falsified by τ is necessary for $\mathcal{F}$.
- Why ?
  - Let $\mathcal{F}'$ be the clauses of $\mathcal{F}$ falsified by τ.
  - Let v be any variable shared among the clauses of $\mathcal{F}'$
    - i.e. $\forall C \in \mathcal{F}'$, $v \in Var(C)$.
  - Remove v from $\mathcal{F}$.
  - All clauses of $\mathcal{F}'$ will be gone, because they all contain v.
  - We get a subformula of $\mathcal{F} \setminus \mathcal{F}'$.
  - But, $\mathcal{F} \setminus \mathcal{F}' \in \mathtt{SAT}$, since τ is its model.
  - Hence, $\mathcal{F} \setminus \mathcal{F}^v \in \mathtt{SAT}$, and so v is necessary for $\mathcal{F}$.
- So, any assignment τ, such that $v \in \bigcap_{C \in Unsat(\mathcal{F},\tau)} Var(C)$, is a *witness of necessity* of v in $\mathcal{F}$.

Optimizations: variable-based model rotation (VMR)

- Note: in VHYB, when $\mathcal{F}_w \setminus \mathcal{F}_w^v \in \mathtt{SAT}$, the assignment returned by the SAT solver (call it τ) *is* a witness of v.

Optimizations: variable-based model rotation (VMR)

- Note: in VHYB, when $\mathcal{F}_w \setminus \mathcal{F}_w^v \in \texttt{SAT}$, the assignment returned by the SAT solver (call it τ) *is* a witness of v.
- τ might be a witness for another variable too. E.g. when $\mathit{Unsat}(\mathcal{F}_w, \tau) = \{C\}$, every variable in C is necessary for $\mathcal{F}_w$.

Optimizations: variable-based model rotation (VMR)

- Note: in VHYB, when $\mathcal{F}_w \setminus \mathcal{F}_w^v \in \texttt{SAT}$, the assignment returned by the SAT solver (call it τ) *is* a witness of v.
- τ might be a witness for another variable too. E.g. when $\mathit{Unsat}(\mathcal{F}_w, \tau) = \{C\}$, every variable in C is necessary for $\mathcal{F}_w$.
- *Variable-based model rotation*: take τ and try to modify it into a witness τ' for another variable. How ?
  1. Flip v to get new assignment τ': all clauses in $\mathit{Unsat}(\mathcal{F}_w, \tau)$ are now satisfied.
  2. But, some other clauses *must* be falsified by τ'.
  3. Any variable shared among the clauses falsified by τ' is necessary.
  4. Continue recursively (but watch for loops).

Optimizations: variable-based model rotation (VMR)

- Note: in VHYB, when $\mathcal{F}_w \setminus \mathcal{F}_w^v \in \texttt{SAT}$, the assignment returned by the SAT solver (call it τ) *is* a witness of v.
- τ might be a witness for another variable too. E.g. when $\mathit{Unsat}(\mathcal{F}_w, \tau) = \{C\}$, every variable in C is necessary for $\mathcal{F}_w$.
- *Variable-based model rotation*: take τ and try to modify it into a witness τ' for another variable. How ?
  1. Flip v to get new assignment τ': all clauses in $\mathit{Unsat}(\mathcal{F}_w, \tau)$ are now satisfied.
  2. But, some other clauses *must* be falsified by τ'.
  3. Any variable shared among the clauses falsified by τ' is necessary.
  4. Continue recursively (but watch for loops).
- Effect: detect multiple necessary variables in a single SAT solver call.

Optimizations: variable-based model rotation (VMR)

- Note: in VHYB, when $\mathcal{F}_w \setminus \mathcal{F}_w^v \in \texttt{SAT}$, the assignment returned by the SAT solver (call it τ) *is* a witness of v.
- τ might be a witness for another variable too. E.g. when $\mathit{Unsat}(\mathcal{F}_w, \tau) = \{C\}$, every variable in C is necessary for $\mathcal{F}_w$.
- *Variable-based model rotation*: take τ and try to modify it into a witness τ' for another variable. How ?
  1. Flip v to get new assignment τ': all clauses in $\mathit{Unsat}(\mathcal{F}_w, \tau)$ are now satisfied.
  2. But, some other clauses *must* be falsified by τ'.
  3. Any variable shared among the clauses falsified by τ' is necessary.
  4. Continue recursively (but watch for loops).
- Effect: detect multiple necessary variables in a single SAT solver call.
- Details and an "extended" (improved) version of VMR are in the paper.

Hybrid VMUS Computation: VMR

Input $\mapsto$ **Output**: Unsatisfiable CNF Formula $\mathcal{F}$ $\mapsto$ $V \in \mathrm{VMUS}(\mathcal{F})$

$V \leftarrow \emptyset$ // VMUS under-approximation
$V_w \leftarrow Var(\mathcal{F})$ // Working (''untested'') set of variables
$\mathcal{F}_w \leftarrow \mathcal{F}$ // Working formula
while $V_w \neq \emptyset$ **do** // Inv: $\mathcal{F}_w = \mathcal{F}|_{V \cup V_w}$ and $\forall v \in V$ is nec. for $\mathcal{F}_w$
  $v \leftarrow \mathtt{PickVariable}(V_w)$
  $V_w \leftarrow V_w \setminus \{v\}$
  $\mathcal{R} \leftarrow \mathtt{CNF}(\neg \mathcal{F}_w^v)$ // Redundancy removal
  $\langle \mathrm{st}, \mathcal{U} \rangle = \mathtt{SAT}(\mathcal{F}_w \setminus \mathcal{F}_w^v)$
  if st = false **then** // v is not necessary for $\mathcal{F}_w$
    $V_w \leftarrow V_w \cap Var(\mathcal{U})$ // Variable-set refinement
    $\mathcal{F}_w \leftarrow \mathcal{F}_w|_{V \cup V_w}$
  else // v is necessary for $\mathcal{F}_w$
    $V \leftarrow V \cup \{v\}$
    $V_w \leftarrow V_w \setminus V$
return V // $V \in \mathrm{VMUS}(\mathcal{F})$ and $\mathcal{F}_w = \mathcal{F}|_V \in \mathrm{VMU}$

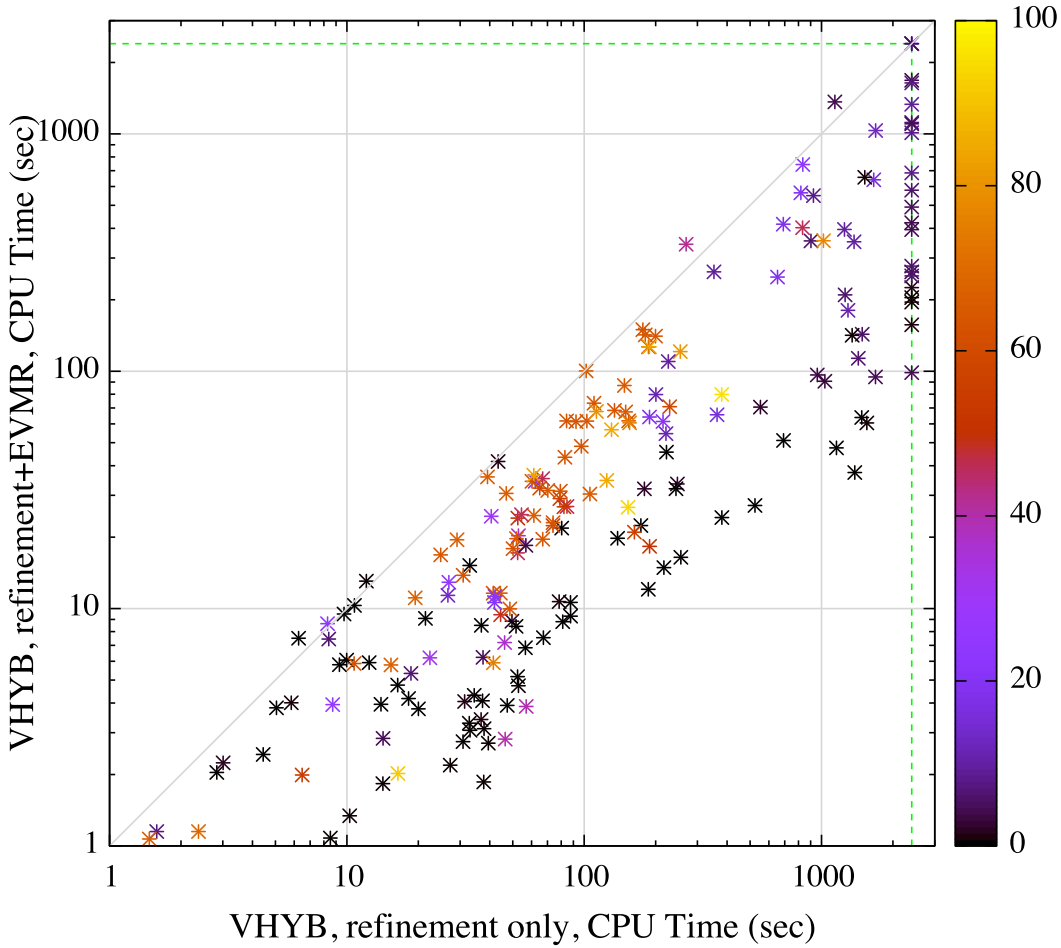
Hybrid VMUS Computation: VMR

Input $\mapsto$ **Output**: Unsatisfiable CNF Formula $\mathcal{F}$ $\mapsto$ $V \in \mathrm{VMUS}(\mathcal{F})$

$V \leftarrow \emptyset$ // VMUS under-approximation
$V_w \leftarrow Var(\mathcal{F})$ // Working (‘‘untested’’) set of variables
$\mathcal{F}_w \leftarrow \mathcal{F}$ // Working formula
while $V_w \neq \emptyset$ **do** // Inv: $\mathcal{F}_w = \mathcal{F}|_{V \cup V_w}$ and $\forall v \in V$ is nec. for $\mathcal{F}_w$
  $v \leftarrow \mathtt{PickVariable}(V_w)$
  $V_w \leftarrow V_w \setminus \{v\}$
  $\mathcal{R} \leftarrow \mathtt{CNF}(\neg \mathcal{F}_w^v)$ // Redundancy removal
  $\langle \mathrm{st}, \mathcal{U}, \tau \rangle = \mathtt{SAT}(\mathcal{F}_w \setminus \mathcal{F}_w^v)$
  if st = false **then** // v is not necessary for $\mathcal{F}_w$
    $V_w \leftarrow V_w \cap Var(\mathcal{U})$ // Variable-set refinement
    $\mathcal{F}_w \leftarrow \mathcal{F}_w|_{V \cup V_w}$
  else // v is necessary for $\mathcal{F}_w$
    $\mathtt{VModelRotation}(\mathcal{F}_w, V, \tau)$ // $v \in V$ after this call
    $V_w \leftarrow V_w \setminus V$
return V // $V \in \mathrm{VMUS}(\mathcal{F})$ and $\mathcal{F}_w = \mathcal{F}|_V \in \mathrm{VMU}$

Impact of variable-based model rotation

- 295 benchmarks from SAT Comp 2011, TO = 1800 sec, MO = 4GB.



- CPU Time, refinement (#sol=239) vs ref.+EVMR (#sol=264)
- Color: VMUS size (% of the number of variables in the input).

Optimizations: redundancy removal

- **Fact:** If $\mathcal{F} \in$ UNSAT, then for any $v \in Var(\mathcal{F})$, $\mathcal{F} \setminus \mathcal{F}^v \in \mathtt{SAT}$ if and only if $(\mathcal{F} \setminus \mathcal{F}^v) \cup CNF(\neg\mathcal{F}^v) \in \mathtt{SAT}$.
  - $\neg CNF(\mathcal{F}^v)$ stands for a satisfiability-preserving transformation of the formula $\neg\mathcal{F}^v$ (we used Plaisted-Greenbaum).
  - During hybrid VMUS extraction: add the clauses of $CNF(\neg\mathcal{F}^v)$ to the formula before SAT solver call.

Optimizations: redundancy removal

- **Fact:** If $\mathcal{F} \in$ UNSAT, then for any $v \in Var(\mathcal{F})$, $\mathcal{F} \setminus \mathcal{F}^v \in \mathtt{SAT}$ if and only if $(\mathcal{F} \setminus \mathcal{F}^v) \cup CNF(\neg\mathcal{F}^v) \in \mathtt{SAT}$.
  - $\neg CNF(\mathcal{F}^v)$ stands for a satisfiability-preserving transformation of the formula $\neg\mathcal{F}^v$ (we used Plaisted-Greenbaum).
  - During hybrid VMUS extraction: add the clauses of $CNF(\neg\mathcal{F}^v)$ to the formula before SAT solver call.
- Effect: make SAT calls easier.

Optimizations: redundancy removal

- **Fact:** If $\mathcal{F} \in$ UNSAT, then for any $v \in Var(\mathcal{F})$, $\mathcal{F} \setminus \mathcal{F}^v \in \mathtt{SAT}$ if and only if $(\mathcal{F} \setminus \mathcal{F}^v) \cup CNF(\neg\mathcal{F}^v) \in \mathtt{SAT}$.
  - $\neg CNF(\mathcal{F}^v)$ stands for a satisfiability-preserving transformation of the formula $\neg\mathcal{F}^v$ (we used Plaisted-Greenbaum).
  - During hybrid VMUS extraction: add the clauses of $CNF(\neg\mathcal{F}^v)$ to the formula before SAT solver call.
- Effect: make SAT calls easier.
- But: if the clauses of $CNF(\neg\mathcal{F}^v)$ are included in the unsatisfiable core (in case of UNSAT outcome), the core cannot be used safely for the variable-set refinement.

Hybrid VMUS Extraction: redundancy removal

Input $\mapsto$ **Output**: Unsatisfiable CNF Formula $\mathcal{F}$ $\mapsto$ $V \in \mathrm{VMUS}(\mathcal{F})$

$V \leftarrow \emptyset$ // VMUS under-approximation
$V_w \leftarrow Var(\mathcal{F})$ // Working (‘‘untested’’) set of variables
$\mathcal{F}_w \leftarrow \mathcal{F}$ // Working formula
while $V_w \neq \emptyset$ **do** // Inv: $\mathcal{F}_w = \mathcal{F}|_{V \cup V_w}$ and $\forall v \in V$ is nec. for $\mathcal{F}_w$
  $v \leftarrow \mathtt{PickVariable}(V_w)$
  $V_w \leftarrow V_w \setminus \{v\}$
  $\mathcal{R} \leftarrow \mathtt{CNF}(\neg \mathcal{F}_w^v)$ // Redundancy removal
  $\langle \mathrm{st}, \mathcal{U}, \tau \rangle = \mathtt{SAT}(\mathcal{F}_w \setminus \mathcal{F}_w^v)$
  if st = false **then** // v is not necessary for $\mathcal{F}_w$
    $V_w \leftarrow V_w \cap Var(\mathcal{U})$ // Variable-set refinement
    $\mathcal{F}_w \leftarrow \mathcal{F}_w|_{V \cup V_w}$
  else // v is necessary for $\mathcal{F}_w$
    $\mathtt{VModelRotation}(\mathcal{F}_w, V, \tau)$ // $v \in V$ after this call
    $V_w \leftarrow V_w \setminus V$
return V // $V \in \mathrm{VMUS}(\mathcal{F})$ and $\mathcal{F}_w = \mathcal{F}|_V \in \mathrm{VMU}$

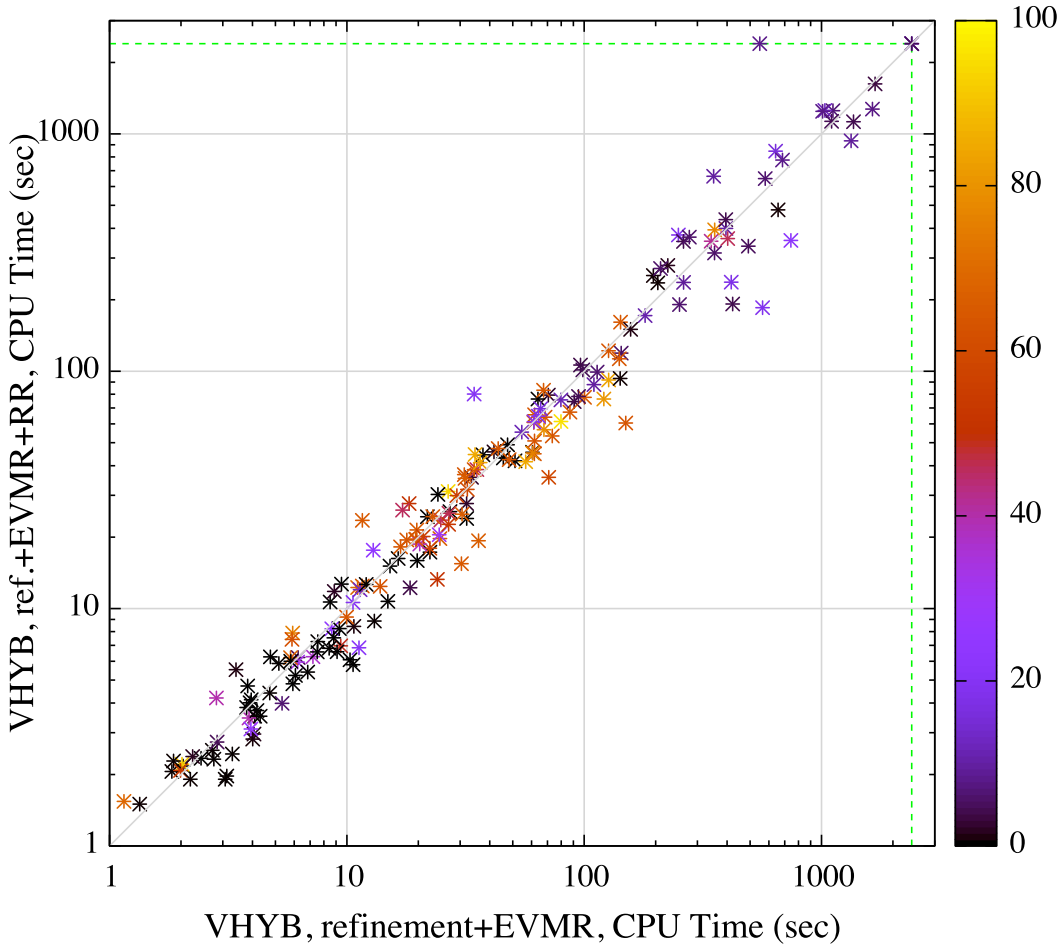
Hybrid VMUS Extraction: redundancy removal

Input $\mapsto$ **Output**: Unsatisfiable CNF Formula $\mathcal{F}$ $\mapsto$ $V \in \mathrm{VMUS}(\mathcal{F})$

$V \leftarrow \emptyset$ // VMUS under-approximation
$V_w \leftarrow Var(\mathcal{F})$ // Working (‘‘untested’’) set of variables
$\mathcal{F}_w \leftarrow \mathcal{F}$ // Working formula
while $V_w \neq \emptyset$ **do** // Inv: $\mathcal{F}_w = \mathcal{F}|_{V \cup V_w}$ and $\forall v \in V$ is nec. for $\mathcal{F}_w$
  $v \leftarrow \mathtt{PickVariable}(V_w)$
  $V_w \leftarrow V_w \setminus \{v\}$
  $\mathcal{R} \leftarrow \mathtt{CNF}(\neg \mathcal{F}_w^v)$ // Redundancy removal
  $\langle \mathrm{st}, \mathcal{U}, \tau \rangle = \mathtt{SAT}((\mathcal{F}_w \setminus \mathcal{F}_w^v) \cup \mathcal{R})$
  if st = false **then** // v is not necessary for $\mathcal{F}_w$
    if $\mathcal{U} \cap \mathcal{R} = \emptyset$ **then** $V_w \leftarrow V_w \cap Var(\mathcal{U})$ // ‘‘Safe’’ refinement
    $\mathcal{F}_w \leftarrow \mathcal{F}_w|_{V \cup V_w}$
  else // v is necessary for $\mathcal{F}_w$
    $\mathtt{VModelRotation}(\mathcal{F}_w, V, \tau)$ // $v \in V$ after this call
    $V_w \leftarrow V_w \setminus V$
return V // $V \in \mathrm{VMUS}(\mathcal{F})$ and $\mathcal{F}_w = \mathcal{F}|_V \in \mathrm{VMU}$

Impact of redundancy removal

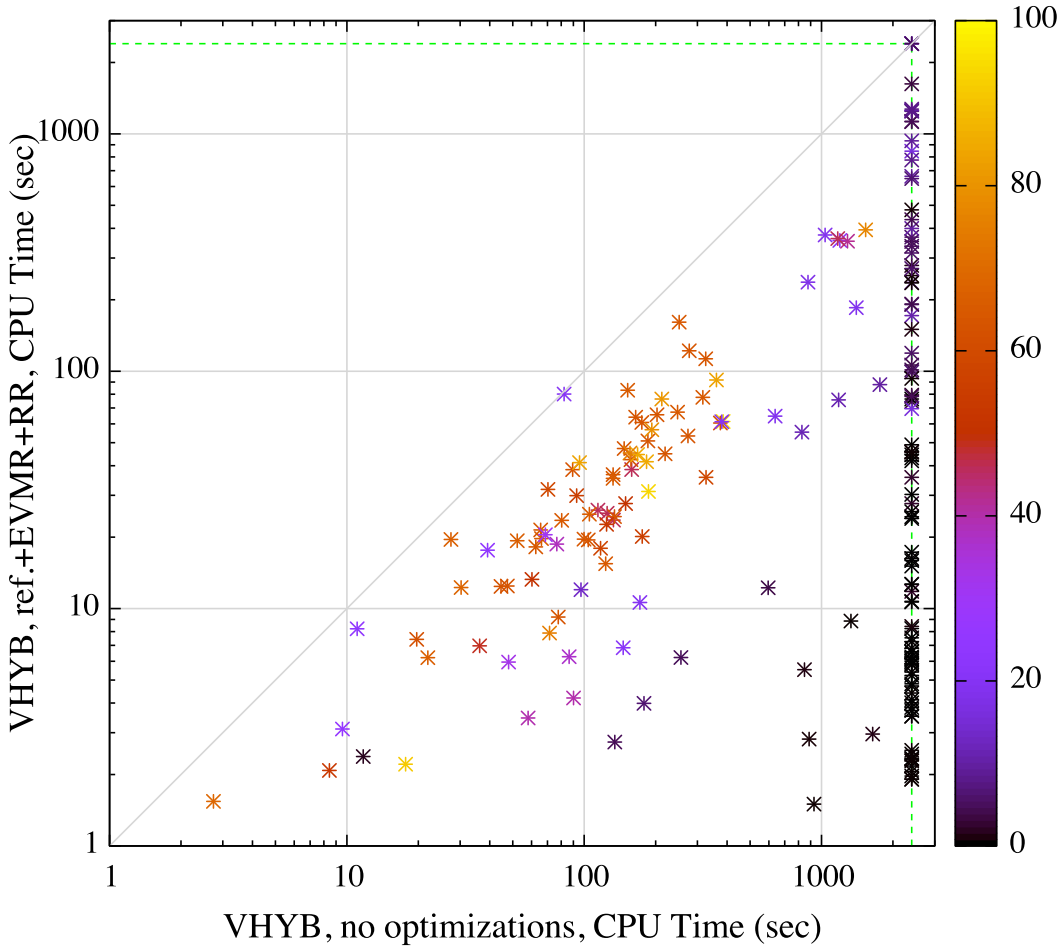
- 295 benchmarks from SAT Comp 2011, TO = 1800 sec, MO = 4GB.



- CPU Time, ref.+EVMR vs ref.+EVMR+RR. Average speedup: 15%.
- Color: VMUS size (% of the number of variables in the input).

Impact of all optimizations

- 295 benchmarks from SAT Comp 2011, TO = 1800 sec, MO = 4GB.



- W/o optimizations solved: 157, avg.vars: 6198, max.vars: 49490.
- With optimizations solved: 265, avg.vars: 105044; max.vars: 1198007.

Additional approaches (in the paper)

Relaxation-variable based approach.

- Lets the SAT solver to *find* a necessary variable.
- Probably won't scale (have not tried it).

Additional approaches (in the paper)

Relaxation-variable based approach.

- Lets the SAT solver to *find* a necessary variable.
- Probably won't scale (have not tried it).

Translation to group-MUS computation problem.

- For dense formulas: for each $v \in Var(\mathcal{F})$ introduce two variables v_p and v_n, and
  1. replace v with v_p and $\neg v$ with v_n;
  2. put the result into group 0;
  3. make a group that says $v_p \leftrightarrow v_n$.

Additional approaches (in the paper)

Relaxation-variable based approach.

- Lets the SAT solver to *find* a necessary variable.
- Probably won't scale (have not tried it).

Translation to group-MUS computation problem.

- For dense formulas: for each $v \in Var(\mathcal{F})$ introduce two variables v_p and v_n, and
  1. replace v with v_p and $\neg v$ with v_n;
  2. put the result into group 0;
  3. make a group that says $v_p \leftrightarrow v_n$.
- For sparse formulas: for each $v \in Var(\mathcal{F})$ introduce an activation variable a_v, and
  1. add an activation variable a_v to clause C if $v \in C$;
  2. put all translated clauses to group 0;
  3. for each variable make a group $\{a_v\}$.

Additional approaches (in the paper)

Relaxation-variable based approach.

- Lets the SAT solver to *find* a necessary variable.
- Probably won't scale (have not tried it).

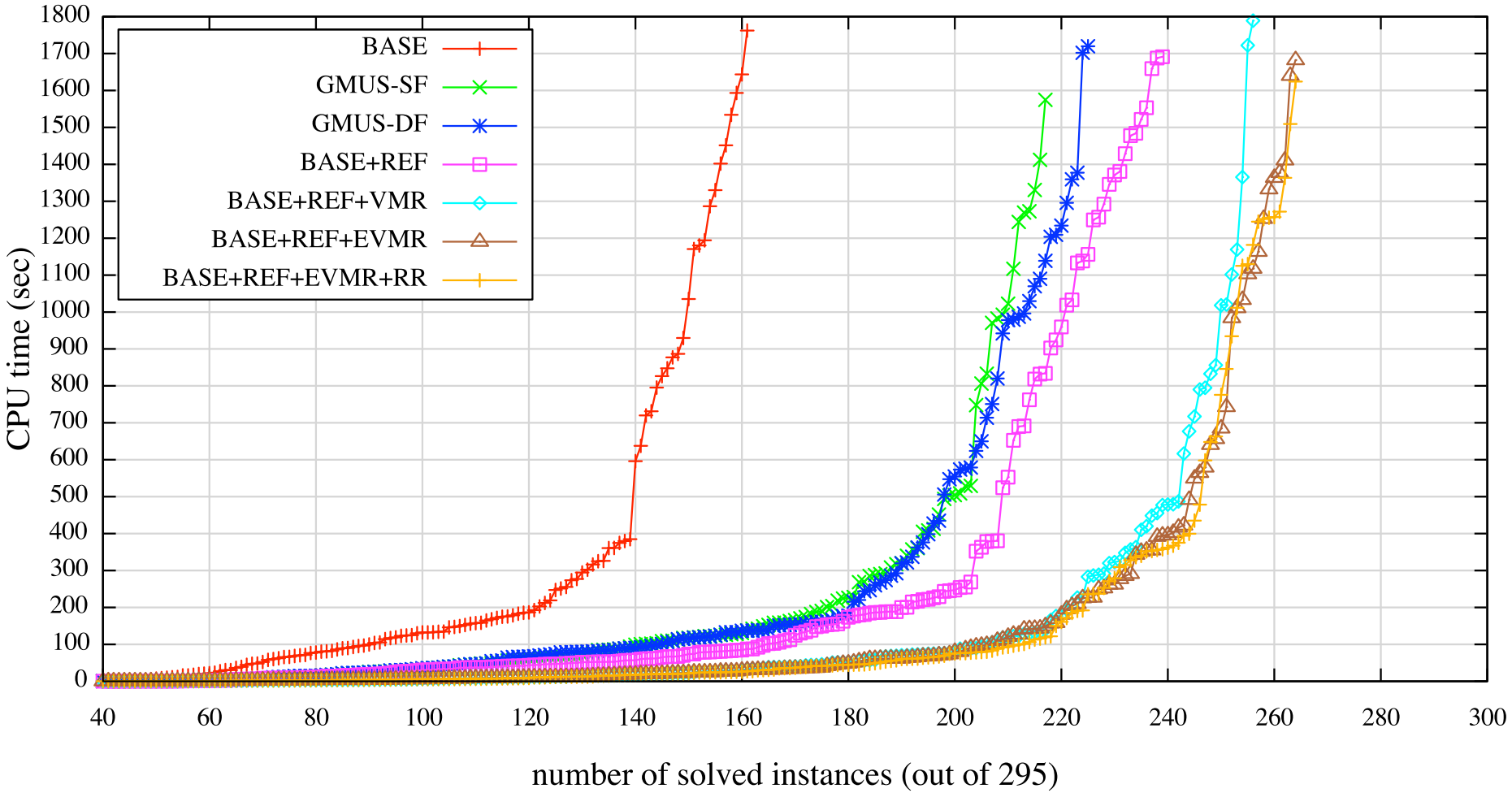
Translation to group-MUS computation problem.

- For dense formulas: for each $v \in Var(\mathcal{F})$ introduce two variables v_p and v_n, and
  1. replace v with v_p and $\neg v$ with v_n;
  2. put the result into group 0;
  3. make a group that says $v_p \leftrightarrow v_n$.
- For sparse formulas: for each $v \in Var(\mathcal{F})$ introduce an activation variable a_v, and
  1. add an activation variable a_v to clause C if $v \in C$;
  2. put all translated clauses to group 0;
  3. for each variable make a group $\{a_v\}$.

Outperformed by Hybrid VMUS algorithm by a wide margin (see paper)

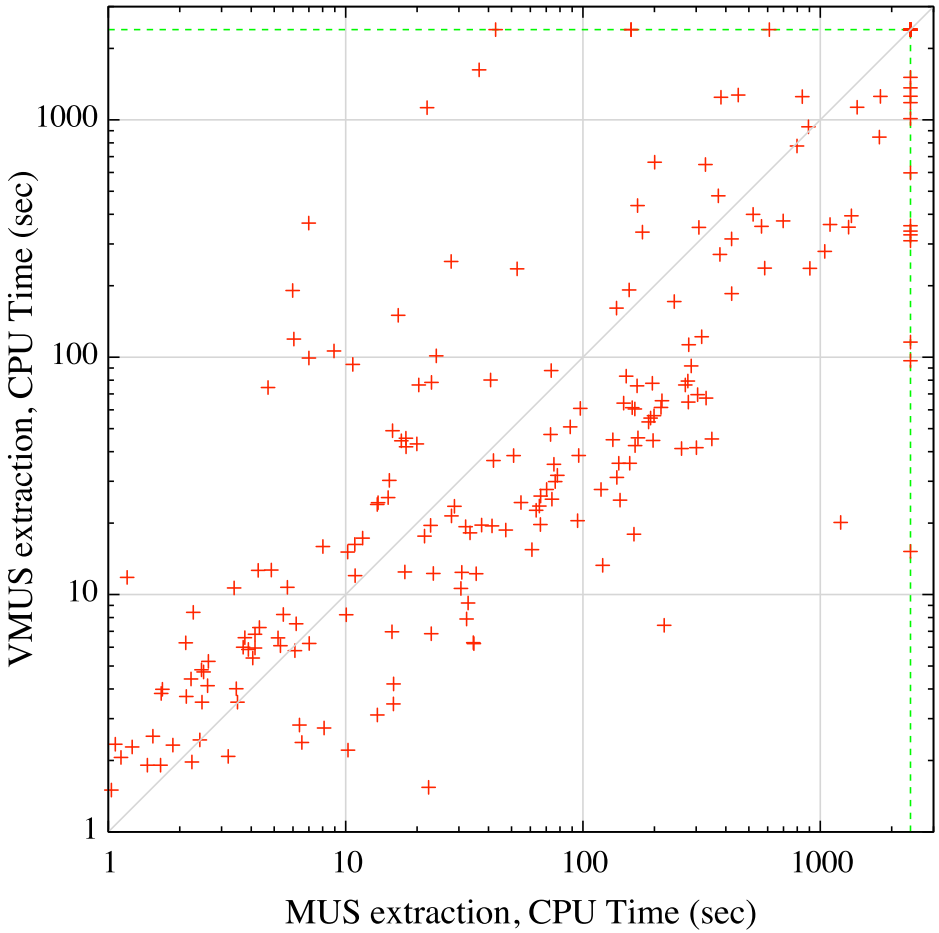
Performance comparison: run-time

- 295 benchmarks from SAT Comp 2011, TO = 1800 sec, MO = 4GB.



VMUS extraction vs MUS extraction

- 295 benchmarks from SAT Comp 2011, TO = 1800 sec, MO = 4GB.



- CPU Time, MUS (#sol=245) vs VMUS extraction (#sol=265).

Summary

- Hybrid VMUS Extraction — significant performance gains vs state-of-the-art due to optimizations.

Summary

- Hybrid VMUS Extraction — significant performance gains vs state-of-the-art due to optimizations.
- Implemented in `MUSer2` (binary distro), at `http://logos.ucd.ie/wiki/doku.php?id=muser`

Summary

- Hybrid VMUS Extraction — significant performance gains vs state-of-the-art due to optimizations.
- Implemented in `MUSer2` (binary distro), at `http://logos.ucd.ie/wiki/doku.php?id=muser`
- A number of alternative approaches (not as efficient).

Summary

- Hybrid VMUS Extraction — significant performance gains vs state-of-the-art due to optimizations.
- Implemented in `MUSer2` (binary distro), at `http://logos.ucd.ie/wiki/doku.php?id=muser`
- A number of alternative approaches (not as efficient).
- VMUS computation is in many cases faster than MUS computation $\Rightarrow$ perhaps some applications of MUSes could be re-considered.

Summary

- Hybrid VMUS Extraction — significant performance gains vs state-of-the-art due to optimizations.
- Implemented in `MUSer2` (binary distro), at `http://logos.ucd.ie/wiki/doku.php?id=muser`
- A number of alternative approaches (not as efficient).
- VMUS computation is in many cases faster than MUS computation $\Rightarrow$ perhaps some applications of MUSes could be re-considered.

Thank you for your attention !